Physics 116C

The differential equation satisfied by Legendre polynomials

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In class we introduced Legendre polynomials through the *generating function*

$$\boxed{g(t,x) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n \,.} \tag{1}$$

The importance of Legendre polynomials in physics is that they satisfy the following differential equation (**Legendre's equation**)

$$\boxed{(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0,} \tag{2}$$

which arises in the solution of many *partial* differential equations.

We wish to show that the $P_n(x)$, defined by Eq. (1), satisfy Eq. (2). However, this requires some boring algebra which I will not cover fully in class. **The details are therefore given in this handout.**

We start by generating a **recurrence relation** between Legende polynomials of different order. To do this we first differentiate Eq. (1) with respect to t:

$$\frac{\partial g(t,x)}{\partial t} = \frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} nP_n(x)t^{n-1} \,. \tag{3}$$

If we multiply by $1-2xt+t^2$ we get

$$(1-2xt+t^2)\sum_{n=0}^{\infty} nP_n(x)t^{n-1} + \frac{t-x}{\sqrt{1-2xt+t^2}} = 0 \,, \tag{4}$$

which, using the generating function, Eq. (1), becomes

$$(1-2xt+t^2)\sum_{n=0}^{\infty} nP_n(x)t^{n-1} + (t-x)\sum_{n=0}^{\infty} P_n(x)t^n = 0 \tag{5}$$

so

$$\sum_m mP_m(x)t^{m-1} - 2x\sum_n nP_n(x)t^n + \sum_s sP_s(x)t^{s+1} + \sum_s P_s(x)t^{s+1} - \sum_n xP_n(x)t^n = 0. \tag{6}$$

If we let $m = n+1$ and $s = n-1$ this becomes

$$\sum_{n=0}^{\infty}\big[(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x)\big]t^n = 0 \,, \tag{7}$$

(for $n = 0$ the $nP_{n-1}(x)$ term is not present).

Since the left hand side must vanish for all t, the coefficient of *each* power of t must vanish, i.e.

$$\boxed{(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)\,,} \tag{8}$$

for $n = 1, 2, 3, \cdots$.

This is the promised *recurrence relation* between Legendre polynomials of different order. It can be used to generate Legendre polynomials of higher order if one knows them of lower order. In particular in we know $P_n(x)$ and $P_{n-1}(x)$ then Eq. (8) gives us $P_{n+1}(x)$. For example, we have

$$P_0(x) = 1; \qquad P_1(x) = x\,, \tag{9}$$

and so, for $n = 1$, Eq. (8) gives

$$2P_2(x) = 3\,x\,x - 1\,, \tag{10}$$

i.e.

$$P_2(x) = \tfrac{1}{2}(3x^2 - 1)\,, \tag{11}$$

which was found earlier, in a more laborious manner, from the generating function. In general, it is easier to determine higher order Legendre polynomials using the recurrence relation, Eq. (8), (and knowledge of low order polynomials) than to expand out the generating function, Eq. (1), to high order.

Additional information is obtained by differentiating Eq. (1) with respect to x:

$$\frac{\partial g(t,x)}{\partial x} = \frac{t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} P_n'(x)t^n\,, \tag{12}$$

which can be rewritten as

$$(1-2xt+t^2)\sum_{n=0}^{\infty} P_n'(x)t^n - \frac{t}{\sqrt{1-2xt+t^2}} = 0\,, \tag{13}$$

or, using the generating function,

$$(1-2xt+t^2)\sum_{n=0}^{\infty} P_n'(x)t^n - t\sum_{n=0}^{\infty} P_n(x)t^n = 0\,. \tag{14}$$

Setting to zero each power of t, as we did to derive Eq. (8), gives

$$\boxed{P_{n+1}'(x) + P_{n-1}'(x) = 2xP_n'(x) + P_n(x)\,.} \tag{15}$$

It is also useful to differentiate Eq. (8) with respect to x. This gives

$$\boxed{(n+1)P'_{n+1}(x) + nP'_{n-1}(x) = (2n+1)[P_n(x) + xP'_n(x)]\,.} \tag{16}$$

We will now use Eqs. (15) and (16) to obtain Legendre's equation, Eq. (2).

Firstly eliminate $P'_n(x)$ between Eqs. (15) and (16), which gives

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)\,. \tag{17}$$

Remembering that we want to get an equation for a single n we subtract Eq. (17) from Eq. (15) and divide by 2 (which eliminates $P'_{n+1}(x)$):

$$P'_{n-1}(x) = -nP_n(x) + xP'_n(x)\,. \tag{18}$$

Similarly we take the sum of Eqs. (15) and (17), and divide by 2, (which eliminates $P'_{n-1}(x)$):

$$P'_{n+1}(x) = (n+1)P_n(x) + xP'_n(x)\,. \tag{19}$$

Next take Eq. (19) with n replaced by $n-1$, and add it to x times Eq. (18) which gives

$$(1-x^2)P'_n(x) = nP_{n-1}(x) - nxP_n(x)\,, \tag{20}$$

which has the advantage that only one term is a derivative. If we differentiate this last equation with respect to x we get

$$(1-x^2)P''_n(x) - 2xP'_n(x) + nP_n(x) + n\big[xP'_n(x) - P'_{n-1}(x)\big] = 0\,. \tag{21}$$

Finally, using Eq. (18), the factor in square brackets is seen to be just $nP_n(x)$, so we obtain the desired result, Eq. (2).