Physics 116C

The differential equation satisfied by Bessel functions

Peter Young

(Dated: October 22, 2009)

In class we introduced Bessel functions through the *generating function*

$$\boxed{g(t,x) = \exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(x) t^n \,.} \tag{1}$$

One reason why Bessel functions are important in physics is that they satisfy the following differential equation (Bessel's equation)

$$\boxed{x^2 J_n''(x) + x J_n'(x) + (x^2 - n^2) J_n(x) \,,} \tag{2}$$

which arises in the solution of several *partial* differential equations in cylindrical polar coordinates.

To show that the $J_n(x)$, defined by Eq. (1), satisfy Eq. (2) requires a bit of boring algebra, which is given in this handout.

First of all differentiate Eq. (1) with respect to t:

$$\frac{\partial g(t,x)}{\partial t} = \frac{x}{2}\left(1 + \frac{1}{t^2}\right) \exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \frac{x}{2}\left(1 + \frac{1}{t^2}\right) \sum_{n=-\infty}^{\infty} J_n(x) t^n \,. \tag{3}$$

But the partial derivative is equal to $\sum_n n t^{n-1} J_n(x)$, and so we have

$$\sum_n n t^{n-1} J_n(x) = \frac{x}{2} \sum_m t^m J_m(x) + \frac{x}{2} \sum_k t^{k-2} J_k(x) \,, \tag{4}$$

where we have deliberately used different names for the summation variables for reasons that will now follow. We define $k = n + 1$ and $m = n - 1$ to get

$$\sum_n n J_n(x) t^{n-1} = \frac{x}{2} \sum_n \left[J_{n-1}(x) + J_{n+1}(x)\right] t^{n-1} \,, \tag{5}$$

which displays explicitly the coefficient of t^{n-1} on both sides of the equation. Equating the coefficients gives

$$\boxed{J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) \,,} \tag{6}$$

which is a *recurrence relation* for the J_n. If we know $J_{n-1}(x)$ and $J_n(x)$, then we can use the recurrence relation to determine $J_{n+1}(x)$.

To obtain the differential equation, Eq. (2), we need to differentiate the generating function, Eq. (1), with respect to x. This gives

$$\sum_n J_n'(x)t^n = \frac{1}{2}\left(t - \frac{1}{t}\right)\sum_n J_n(x)t^n. \tag{7}$$

Following the same method as we used to get Eq. (5) we can display explicitly the coefficients of t^n on both sides of the equation and equate them. This gives

$$\boxed{J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x)\,.} \tag{8}$$

As a special case, consider $n = 0$. Since we already know that $J_{-1}(x) = -J_1(x)$, Eq. (8) gives

$$\boxed{J_0'(x) = -J_1(x)\,.} \tag{9}$$

To proceed we add Eqs. (6) and (8) and divide by 2, which gives

$$xJ_n'(x) = xJ_{n-1}(x) - nJ_n(x)\,, \tag{10}$$

which has the advantage that it only involves two values of n. Similarly, we subtract Eq. (8) from Eq. (6) and divide by 2, which gives

$$xJ_n'(x) = -xJ_{n+1}(x) + nJ_n(x)\,. \tag{11}$$

It is sometimes useful to rewrite Eqs. (10) and (11) in a different way as follows. Equation (10) can be written

$$xJ_n'(x) + nJ_n(x) = xJ_{n-1}(x) \tag{12}$$

and multiplying by x^{n-1} gives

$$\boxed{\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x)\,.} \tag{13}$$

Similarly Eq. (11) can be written

$$\boxed{\frac{d}{dx}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n-1}(x)\,.} \tag{14}$$

To obtain Bessel's equation we differentiate Eq. (10) w.r.t. x which gives

$$xJ_n''(x) + (n+1)J_n'(x) - nJ_n'(x) - xJ_{n-1}'(x) = 0\,. \tag{15}$$

[Eq. (15) times x] minus [Eq. (10) times n] gives

$$x^2 J_n''(x) + xJ_n'(x) - n^2 J_n(x) - x^2 J_{n-1}'(x) + (n-1)xJ_{n-1}(x) = 0. \tag{16}$$

This almost gives the desired result, Eq. (2), but we need to rearrange the last two terms in terms of J_n rather than J_{n-1}. This can be done by taking Eq. (11) with n replaced by $n-1$, i.e.

$$(n-1)J_{n-1}(x) - xJ'_{n-1}(x) = xJ_n(x)\,. \tag{17}$$

Substituting this into Eq. (16) gives Bessel's equation

$$\boxed{x^2 J''_n(x) + xJ'_n(x) + (x^2 - n^2)J_n(x)\,,} \tag{18}$$

as desired.

We should point out that one can also verify Bessel's equation starting from the *series expansion* definition of Bessel functions that we discussed in class.