Physics 116C

The orthogonality relation satisfied by Bessel functions

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We showed in class that the Bessel function $J_p(x)$ satisfies the following differential equation (Bessel's equation)

$$x^2\frac{d^2J_p}{dx^2}+x\frac{dJ_p}{dx}+(x^2-p^2)J_p=0\,. \tag{1}$$

which can be written as

$$\boxed{x\frac{d}{dx}\left(x\frac{dJ_p}{dx}\right)+(x^2-p^2)J_p=0\,.} \tag{2}$$

The variable p need not be an integer. It turns out to be useful to define a new variable t by $x = at$, where a is a constant which we will take to be a zero of J_p, i.e. $J_p(a) = 0$. Let us define

$$u(t)=J_p(at)\,, \tag{3}$$

which implies

$$u(1)=0\,, \tag{4}$$

and substituting into Eq. (2) gives

$$t\frac{d}{dt}\left(t\frac{du}{dt}\right)+(a^2t^2-p^2)u=0\,, \tag{5}$$

since xd/dx is equivalent to td/dt. We can also write down the equation obtained by picking another zero, b say. Defining

$$v(t)=J_p(bt)\qquad\text{so}\qquad v(1)=0\,, \tag{6}$$

we have

$$t\frac{d}{dt}\left(t\frac{dv}{dt}\right)+(b^2t^2-p^2)v=0\,. \tag{7}$$

To derive the orthogonality relation, we multiply Eq. (5) by v, and Eq. (7) by u. Subtracting and dividing by t gives

$$v\frac{d}{dt}\left(t\frac{du}{dt}\right)-u\frac{d}{dt}\left(t\frac{dv}{dt}\right)+(a^2-b^2)tuv=0\,. \tag{8}$$

The first two terms in Eq. (8) can be combined as

$$\frac{d}{dt}\left(v\,t\,\frac{du}{dt} - u\,t\,\frac{dv}{dt}\right), \tag{9}$$

since the extra terms present in Eq. (9), but not in Eq. (8), when the derivatives are expanded out are equal and opposite and so cancel. Hence we have

$$\frac{d}{dt}\left(v\,t\,\frac{du}{dt} - u\,t\,\frac{dv}{dt}\right) + (a^2 - b^2)\,t\,u\,v = 0\,. \tag{10}$$

We next integrate this over the range of t from 0 to 1, which gives

$$\left[v\,t\,\frac{du}{dt} - u\,t\,\frac{dv}{dt}\right]_0^1 + (a^2 - b^2)\int_0^1 t\,u(t)\,v(t)\,dt = 0\,. \tag{11}$$

The integrated term vanishes at the lower limit because $t = 0$, and it also vanishes at the upper limit because $u(1) = v(1) = 0$, see Eqs. (4) and (6). Hence, if $a \neq b$, Eq. (11) gives

$$\int_0^1 t\,u(t)v(t)\,dt = 0\,, \tag{12}$$

which, using Eqs. (3) and (6), can be written

$$\boxed{\int_0^1 t\,J_p(at)J_p(bt)\,dt = 0\,.} \tag{13}$$

This is the desired orthogonality equation. Remember we require that a and b are distinct zeroes of J_p, so both Bessel functions in Eq. (13) vanish at the upper limit.

If $a = b$ you showed in a homework problem that the corresponding integral is given by

$$\int_0^1 t\,J_p^2(at)\,dt = \frac{1}{2}J_p'^{\,2}(a)\,, \tag{14}$$

where $J_p'(a) \equiv dJ_p(x)/dx|_{x=a}$. This can be written in different ways. From Eq. (10) of the handout on "The differential equation satisfied by Bessel functions" we have $J_p'(a) = J_{p-1}(a)$ (remember that $J_p(a) = 0$), and from Eq. (11) we have $J_p'(a) = -J_{p+1}(a)$. Hence

$$\boxed{\int_0^1 t\,J_p^2(at)\,dt = \frac{1}{2}J_p'^{\,2}(a) = \frac{1}{2}J_{p-1}^2(a) = \frac{1}{2}J_{p+1}^2(a)\,,} \tag{15}$$

where a is a zero of $J_p(x)$, i.e. $J_p(a) = 0$.

We are now able to expand a given function $f(x)$ in the interval from zero to 1 (provided $f(1) = 0$) as a Bessel series

$$\boxed{f(x) = \sum_m a_m J_p(c_{mp}x)\,,} \tag{16}$$

where c_{mp} is the $m-$th zero of the Bessel function $J_p(x)$. Note that p, the order of the Bessel function, is fixed in Eq. (16). Equation (16) will be very useful when solving partial differential equations with certain boundary conditions. Multiplying Eq. (16) by $x J_p(c_{np}x)$, integrating from 0 to 1, and using Eqs. (13) and (15), the coefficient a_n is easily seen to be

$$\boxed{a_n = \frac{1}{\frac{1}{2} J_p'^2(c_{np})} \int_0^1 x\, f(x)\, J_p(c_{np}x)\, dx\,.} \tag{17}$$

From Eq. (15) we see that $J_p'(c_{np})$ in the denominator can be replaced by $J_{p-1}(c_{np})$ or $J_{p+1}(c_{np})$.

Bessel series are analogous to Fourier series and Legendre series that we have met before. As we shall discuss in class and in more detail in the homework, Bessel functions, Legendre polynomials, and sines and cosines, are just particular examples of sets of functions which solve a general class of differential equations known as “Sturm-Liouville”. Sturm Liouville equations are important because:

> *All equations of the Sturm-Liouville type have an orthogonality property which permits a given function defined over an appropriate range and with appropriate boundary conditions to be expressed as a linear combination of their solutions, in which each coefficient can be determined simply by doing an integral.*

You will investigate Sturm-Liouville equations in a problem in the next homework assignment.