Physics 116C

Some comments on contour integrals

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I. INTEGRALS OF THE TYPE $\int_{-\infty}^{\infty} f(x)\,dx$

We frequently wish to evaluate integrals of the form

$$\boxed{I = \int_{-\infty}^{\infty} f(x)\,dx\,,} \tag{1}$$

in which the function f has only poles in the complex plane. In this section we will see that, in many cases, I can be determined from the integral

$$\oint_C f(z)\,dz$$

where C is the the semi-circular contour in the complex plane shown in Fig. 1, in the limit $R \to \infty$.

Clearly

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\text{semi}} f(z)\,dz$$

where the first term is the part along the real axis which, for $R \to \infty$, is the desired integral I in Eq. (1), and the second term is the integral around the semicircle. The contour integral $\oint_C f(z)\,dz$ is evaluated using the residue theorem ($2\pi i$ times the sum of the residues of $f(z)$ inside C). We presume we are able to calculate this. If the contribution from the semicircle vanishes for $R \to \infty$

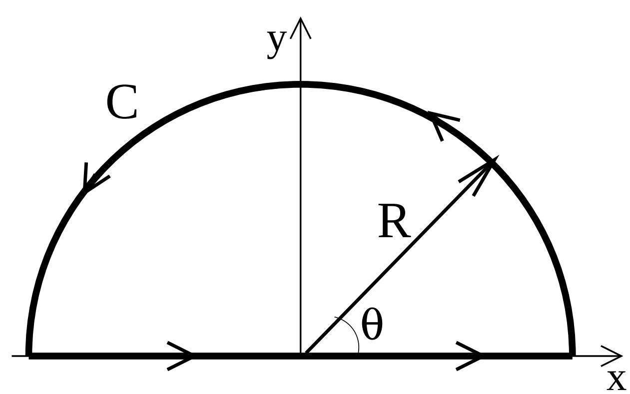

FIG. 1: Contour completed in the upper half-plane.

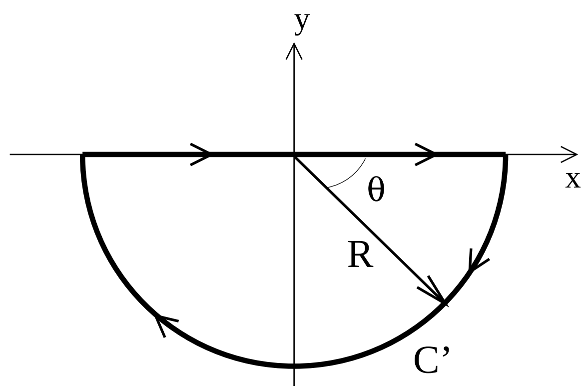

FIG. 2: Contour completed in the lower half-plane.

we see that I is just equal to the contour integral. We therefore need to know under what conditions the contribution from the semicircle vanishes for $R \to \infty$.

Here we give a rough argument. The integral is the length of the semicircle times the average value of $f(z)$ along the semicircle. The length of the semicircle is πR. Hence, if $|f(z)|$ vanishes faster than $1/R$ (*i.e.* faster than $1/|z|$) for $|z| \to \infty$ then the contribution to the integral from the semicircle should vanish. More precise derivations of this result are given in the books. The conclusion is therefore that

$$\boxed{\text{if } |f(z)| \text{ tends to zero faster than } 1/|z| \text{ for } |z| \to \infty}$$

then we can neglect the contribution from the "semicircle at infinity" and so $I = \int_{-\infty}^{\infty} f(x)\, dx$ is given by

$$\boxed{I = 2\pi i \,(\text{sum of residues of } f(z) \text{ in upper half plane}).} \tag{2}$$

Under the same conditions for the vanishing of the contribution from the semicircle, one can alternatively complete the contour by a semicircle in the lower half-plane. This is shown as C' in Fig. 2. In this case the contour integral is equal to

$$\boxed{I = -2\pi i \,(\text{sum of residues of } f(z) \text{ in lower half plane}),} \tag{3}$$

where the minus sign is because the contour is traversed in a clockwise manner. We emphasize that if $f(z)$ has only poles and vanishes faster than $1/|z|$ as $|z| \to \infty$, one can use *either* Eq. (2) or Eq. (3) to determine I in Eq. (1).

As an example consider

$$I = \int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx\,. \tag{4}$$

This is a simple case which can be evaluated by conventional means:

$$I = \left[\tan^{-1} x\right]_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \boxed{= \pi\,.} \tag{5}$$

We will now rederive this result by using a contour integral. Of course, the reason we study contour integrals is that this technique can be used for more complicated examples which can not be evaluated by standard techniques.

The poles of

$$f(z) = \frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)}$$

are at $z = i$ and $z = -i$. The residue at $z = i$ is $1/(z+i)|_{z=i} = 1/(2i)$. Similarly the residue at $z = -i$ is $-1/(2i)$.

Since $|f(z)| \to 1/|z|^2$ for $z \to \infty$, which is faster than $1/z$, the contribution from the semicircle at infinity (in either the upper or lower half-plane) is negligible. Completing in the upper half-plane encloses the pole at $z = i$ which has residue $1/(2i)$. Hence, by Cauchy's residue theorem, Eq. (2), we have

$$I = 2\pi i \,\frac{1}{2i} = \boxed{\pi\,,}$$

in agreement with Eq. (5), which was found by direct methods.

Completing the contour in the lower half-plane, see Fig. 2, we pick up the contribution from the pole at $z = -i$, which has residue $-1/(2i)$. Since the contour in Fig. 2 is traversed in a clockwise manner, there is an additional minus sign, and so, according to Eq. (3), we have

$$I = -2\pi i \left(-\frac{1}{2i}\right) = \boxed{\pi\,,}$$

as before. Hence integrals of the type in Eq. (1) can be done by completing the contour in either the upper or lower half-plane.

II. JORDAN'S LEMMA

Sometimes (especially when doing Fourier transforms) one needs to evaluate integrals of the form

$$\boxed{I_2 = \int_{-\infty}^{\infty} e^{ikx} f(x)\, dx\,,} \tag{6}$$

where again the function f has only poles in the complex plane. We will take k to be real but allow for it to have either sign.

Again we evaluate this from a contour integral, in this case

$$\oint e^{ikz} f(z)\, dz\,,$$

where the contour will be either the contour C in the upper half-plane, Fig. (1), or the contour C' in the lower half-plane, Fig. 2. We will see that we have to choose one or the other depending on the sign of k.

Along the semicircle in the upper half-plane, Fig. 1, we write $x = R\cos\theta, y = R\sin\theta$, so $ikz = ik(x+iy) = ikR\cos\theta - kR\sin\theta$, *i.e.*

$$e^{ikz} = e^{ikR\cos\theta} e^{-kR\sin\theta}\,.$$

If $k > 0$ this is is *very small* along the semicircle in the upper half-plane (except for short segments near the real axis where θ is very small) because of the real (decaying) exponential $e^{-kR\sin\theta}$. Clearly e^{ikz} being small *helps* convergence relative to the case studied in the previous section where there was no factor of e^{ikz}. In fact, as shown in the books, one only needs $|f(z)|$ to *vanish* for $|z| \to \infty$ (rather than vanish faster than $1/|z|$) in order for the contribution from the semicircle to vanish for $R \to \infty$. This is known as **Jordan's Lemma**.

To summarize, for $k > 0$,

$$\boxed{\text{if } |f(z)| \text{ tends to zero for } |z| \to \infty}$$

then we can neglect the contribution from the "semicircle at infinity" and so $I_2 = \int_{-\infty}^{\infty} e^{ikx} f(x)\, dx$ is given by

$$\boxed{I_2 = 2\pi i \times (\text{sum of residues of } f(z)e^{ikz} \text{ in the upper half plane}) \qquad (k > 0),} \tag{7}$$

If $k < 0$, we complete the contour by a semicircle in the lower half-plane, Fig. 2, in which case the contribution from the semicircle is again negligible, and so

$$\boxed{I_2 = -2\pi i \times (\text{sum of residues of } f(z)e^{ikz} \text{ in the lower half plane}) \qquad (k < 0),} \tag{8}$$

Again the minus sign is because the contour C' is traversed in a clockwise sense. We emphasize that if $f(z)$ has only poles and $|f(z)| \to 0$ as $|z| \to \infty$, then the integral I_2 in Eq. (6) can be determined from Eq. (7) if $k > 0$ and Eq. (8) if $k < 0$.

If k is complex, similar considerations show that we complete the contour in the upper half-plane if $\mathrm{Re}(k) > 0$ and in the lower half-plane if $\mathrm{Re}(k) < 0$.

As an example consider

$$I_2(k) = \int_{-\infty}^{\infty} \frac{e^{ikx}}{a^2 + x^2}\, dx \qquad (a > 0). \tag{9}$$

Firstly we note that $I_2(k)$ is a real even function of k. To see this write $e^{ikx} = \cos kx + i \sin kx$ and note that the imaginary part of I_2 is zero because $\sin kx$ is an odd function of x (and $1/(a^2 + x^2)$ is even), and the real part of I_2 is an even function of k because $\cos kx$ is an even function of k.

To evaluate $I_2(k)$ for $k > 0$ we complete in the upper half-plane, Fig. 1. With $f(z) = 1/(a^2+z^2)$, we have $|f(z)| \to 0$ for $|z| \to \infty$ and so, according to Jordan's Lemma, the contribution from the semicircle vanishes for $R \to \infty$. We pick up the contribution from the pole of $e^{ikz}/(a^2 + z^2)$ at $z = ia$ where the residue is $e^{-ka}/(2ia)$. Hence, from Eq. (7)

$$I_2(k) = 2\pi i \, \frac{e^{-ka}}{2ia} = \frac{\pi}{a}\, e^{-ka} \qquad (k > 0)\,. \tag{10}$$

For $k < 0$ we complete in the lower half-plane, Fig. 2. The residue of the pole at $z = -ia$ is $e^{ka}/(-2ia)$. Hence, from Eq. (8)

$$I_2(k) = -2\pi i \left(-\frac{e^{ka}}{2ia} \right) = \frac{\pi}{a}\, e^{ka} \qquad (k < 0)\,. \tag{11}$$

We can conveniently combine Eqs. (10) and (11) as

$$\boxed{I_2(k) = \frac{\pi}{a}\, e^{-|k|a}\,,} \tag{12}$$

which is valid for either sign of k. We see that $I_2(k)$ is a real even function of k, as noted earlier. Note, too, that for $a = 1, k = 0$, Eq. (12) reduces to Eq. (5), as it should.

The integral in Eq. (4) is easy to evaluate using using both standard methods and contour integration. However, the integral in Eq. (9) is difficult using standard methods, and all the derivations of it that I have seen in books used a contour integral.