Calculus I lecture notes

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1 Lecture I Appendix A, B and Section 1.1

1.1 Appendix I, II: The line and plane

These appendices are very basic, and they should be review. You will be required to read any sections that are not covered in class. One thing that you might not be overly familiar with are inequalities that involve absolute value and quadradic relationships. For example, before the end of this lecture, I hope to tell you how to find for what values of x does $2\,x^2 + x \leq 1$ hold and what values of x does $5 \leq |x+1|$ hold.

1.1.1 Basic Set Theory

Before getting into that, I will review some set theoritic notation including union, intersection and interval notation. Informally, a set is a ollection objects. It is probably the most important concept in all of mathematics. A set could be the set of days in a weekend

$$\{ \text{ Saturday, Sunday } \}$$

or it could be empty as in the set of days of the week not ending in the letter "y".

The primary set that we will be concerned with in this course is the set $\mathbb{R}$ or real numbers. This set contains the intergers the rationals and the irrationals. Irrational numbers are numbers that can not be written as a quotient of integers. They include π, e, and the square root of any nonperfect square such as 2, 3, 5, 7, 8 .... Of primary concern in $\mathbb{R}$ are the intervals. There are several forms of intervals depending on whether or not the interval contains its endpoints. The two basic intervals are called the closed and open interval and are denoted by $[a,b]$ and (a,b) respectively. Where $[a,b]$ means the set of points x on the real line such that x is greater than or equal to a and x is less than or equal to b, and (a,b) means the set of x such that x is strictly greater than a and strictly less than b. In symbols we write

$$[a,b] = \{x : a \leq x \leq b\}$$
$$(a,b) = \{x : a < x < b\}$$

similiarly, we can define half open and half closed intervals for example as

$$[a,b) = \{x : a \leq x < b\}.$$

The last point about set theoritical constructions that merits review is that of union and intersection. The union of two intervals (or two sets in general) is the set formed by "sticking" the sets together, and the symbol $\bigcup$ is used. So for instance the set $[a,b]\bigcup(c,d]$ is the set of real numbers that are either greater than or equal to a and less than or equal to b OR greater than or equal to c and less than or equal to d.

The intersection, on the other hand, is the set of things the two sets have in common and is denoted by $\bigcap$. For instance the set $[a, b] \bigcap (c, d]$ is the set of real numbers that are either greater than or equal to a and less than or equal to b AND greater than or equal to c and less than or equal to d. Notice that the only difference between the constuctions is the use of the word 'or' in the first and 'and' in the second.

1.1.2 Solving Inequalities Involving Absolute Values

The absolute value of a number a is defined to be the distance of a to the origin. It is denoted by $\|a\|$. In other words, the absolute value is deined as:

$$\boxed{\begin{array}{c} \|a\| = -a \text{ if } a < 0 \\ \|a\| = a \text{ if } a \geq 0 \end{array}}$$

Probably the most unfamiliar type of problem on this homework is problem 45 on page A9. I will do an example similiar to this problem:

Example 1 Find the set of points x so that $|3\,x + 1| = |x + 4|$ holds. Now, from the definition of the absolute value, there are at most four cases to consider:

1. Both terms are negative
2. The left term is positive and the right term is negative
3. The left term is negative and the right term is positive
4. Both terms are positive.

However when we look closer, in the first case we need to consider the equation $-3\,x - 1 = -x - 4$ which is the same thing as $3\,x + 1 = x + 4$ when the negative sign is canceled from both sides. However, this equation is exactly the same as the one that we need to consider in the forth scenerio. Similiarly, the second and third equations are really the same thing as well. Thus we are looking for the set of points x so that LHS = RHS or LHS = -RHS where LHS means the left hand side of the equation and RHS means the right hand side of the equation in our example. Moreover, we notice that if either of these situations takes place, we have that $|LHS| = |RHS|$. So, proceeding with our concrete example x satisfies:

$$\begin{aligned} |3\,x + 1| &= |x + 4| && (1) \\ \text{if and only if x satisfies } 3\,x + 1 &= x + 4 && (2) \\ \text{or } 3\,x + 1 &= -x - 4. && (3) \end{aligned}$$

solving these two equations should be familiar to you from your basic high school algebra. We see that our solutions are $x = 1$ or $-5/4$.

$\square$

1.1.3 Solving inequalities involving quadratic terms

It is also possible that you have yet to have seen a question along the lines of:

Example 2 Find the set of points x so that $x^2 - x \leq 2$. Again, the general technique for solving such a problem is essentially standard. There are several tricks to solving problems such as these, but I think that these problems are simple enough to use a straight foward approach. Such an approach has the advantage over a trick because it makes it easier to remember such an approach for an exam or quiz. It is also more transparent in the nature of its inner workings. Here's a general procedure for solving such problems:

1. Bring everything to one side so the equation is in the form $A \leq 0$ where A is some polynomial.
2. Factor the polynomial on the left into linear terms. All polynomials in this assignment will factor into such linear terms (other wise the equation would have complex roots which would indicate that the polynomial is always greater than or always less than 0).
3. Use the logic that for a product of terms is less than zero if and if one of the factors is negative (but not both).

Let's apply this general procedure to our specific problem.

$$x^2 - x \leq 2 \tag{4}$$

$$x^2 - x - 2 \leq 0 \tag{5}$$

$$(x-2)(x+1) \leq 0. \tag{6}$$

Now comes the time where we need to think a little bit about what we have. From the final inequality above, we know that we need to have $x - 2 \leq 0$ and $x + 1 \geq 0$ or $x - 2 \geq 0$ and $x + 1 \leq 0$. Looking closer at the second statement this would imply that x is less than or equal to -1 and x is greater than or equal to 2. However, this of course is impossible. Thus, we are required that $x \leq 2$ and $x \geq -1$. In the notation developed above, this means that our solution set is $[-1, 2]$.

solving these two equations should be familiar to you from your basic high school algebra. We see that our solutions are $x = 1$ or $-5/4$.

□

1.1.4 Appendix B: Cartesian geometry

This section should be completely review. Topics covered in this section include finding the slope of a line and writing the equation of a line when given two points on that line. I will quickly review how to find the slope of a line and how to compute the equation of a line in the plane. First, let's recall a basic statement about straight lines.

Lines in the plane Every line in the plane is uniquely determined by its value at two points.

What this means is if we have to lines l_1 and l_2, and both lines pass through two different points, then the lines are actually the same.

1.2 Chapter I section 1: Four ways to represent a function

After touching on some of the main concepts presented in the appendices , we now begin our study of calculus. We begin with the central concept in mathematics. That of a function. A function is nothing more than a explicit way of expressing a particular kind of relation between two sets. I will give a slightly less fluffy introduction to what a function is than the textbook does. I think that this is important because most students do not understand the difference between statements such as $y = x^2$ and $f(x) = x^2$. Techniqually, a funtion f has three ingredients.

1. A set X called the **domain** of the function.

2. A set Y called the **codomain** of the function.
3. A rule.

This is written: $f : X \to Y$ The rule must satisfy two conditions:

1. For every value x in the domain there is an element y of the codomain so that the rule relates the value x to the value y.
2. If the rule relates x both of the elements y_1 and y_2 of the codomain then $y_1 = y_2$.

If the rule f assigns x to y we use the familiar notation $y = f(x)$. Notice that while we require that every value of the domain be used, we do not require every value of the codmain be used. The set of values used in the codomain is known as the **range**. In this class since the codomain can be any set containing the range, we will mostly only concern ourselves with the range. However, in most other math courses the differences between the codomain and the range are very important. For example, the rule that sends x to $frac(1, x)$ is not a function from $\mathbb{R}$ to $\mathbb{R}$ since it does not take a value at $x = 0$, however the rule that takes x to $\sin(x)$ is a function from $\mathbb{R}$ to $\mathbb{R}$ even through this function never takes any values above 1 or below -1. We say that the sin function has range $[-1, 1]$.

In calculus the domain and codomain (and hence the range) will almost always be subsets of $\mathbb{R}$, the real numbers. Opposed to in higher calculus classes where the domains and ranges will be subsets of $\mathbb{R}^n$ for higher values of n. In graduate courses on manifolds, the domains and ranges in which one does calculus are objects such as balls and donughts. However, a function can be a much more primative object. For example, the rule that assigns a moment in time to the day of the week the moment occurs is a function since every moment occurs in one and only one day of the week. However, the rule that takes a day of the week and assigns to it the moments that occur during that day is not a function. The typical way a function is described in highschool is as a machine that takes in values of the domain and spits out values in the range.

Example 3

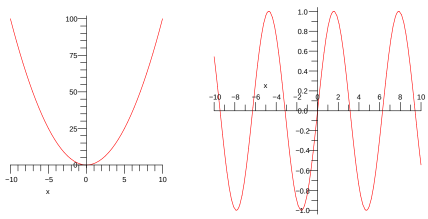

Figure 1: The graph of the functions $y = x^2$ and $y = \sin(x)$

The primary way a function is depicted in this course is as a graph (see examples above). In later sections we will describe methods for graphing functions that we encounter. A graph is defined as the set of points $\{x, f(x) : \text{x is in the domain of the function } f\}$. We can represent this as a subset of the plane by placing a point for every point above.

Conversly, given a subset of the plane, it is easy to tell whether or not the subset is the graph of some function f. This is done via the so-called vertical line test. The first condition to be a function is easily satisfied by simply letting the domain be the set of points where the function is defined. The second however does need to be verified. This can be done by making sure that for every value of x (values along the x-axis) drawing a line that is perpindicular to the axis. If for every x this line only hits *one* point of the graph, we have a function. Summarizing:

The Vertical Line Test A subset of the xy-plane is the graph of some function of x if and only if no vertical line intersects the the subset more than once.

Example 4

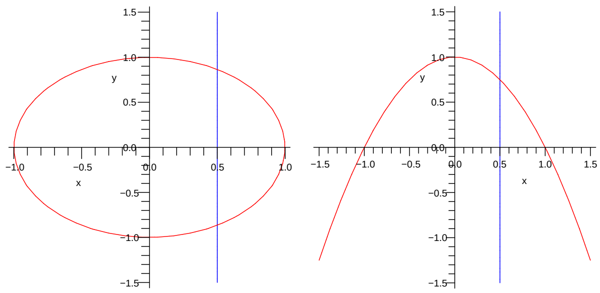

Figure 2: The first subset of the plane fails the vertical line test and the second passes. Thus the first is not the graph of any function while the second one is (namely the function $f(x) = 1 - x^2$).

2 Lecture II Section 1.1 and 1.2

Today we finish our discussion of section 1.1, and we begin our discussion of section 1.3. We will skip section 1.2, and we will not return to it later. Today our focus will be entirely on real valued functions. We will compute the domain and ranges of several examples, and we will go on to describe ways of making new functions from old ones.

2.1 Chapter 1 Section 1 Continued

I will first give you an example. The example is one of your assigned problems, and it is exercise 24 on page 23.

Example 1 Compute the domain of the function

$$\frac{5x+4}{x^2+3x+2}$$

The first thing to note about this problem is that it is not properly worded. The wording should be to find the "maximal domain" of the function. However, this poor word choice is typical, and you should understand what they mean by such statements. We know that our maximal domain will be a subset of $\mathbb{R}$. We want to know the mimimal number of points that we need to eliminate from the domain. To do this we factor the denominator $x^2 + 3x + 2$ into it's linear terms, $(x + 2)(x + 1)$. From this factorization, we know that the function is not be defined at $x = -2$ and at $x = -1$. Thus, these points can not be in

the domain of the function (remember, by definition of a function *every* point of the domain must have a unique element that it gets mapped to. Thus our maximal domain is $(-\infty,-2)\bigcup(-2,-1)\bigcup(-1,\infty)$.
$\square$

2.1.1 Piecewise Defined Functions

The next topic is also a topic that should not be new to any of the students here, but it will possibly require a couple of examples to clarify. Now, there is no reason to expect that our real valued functions actually have a single formula to compute them (or any formula at all!). Often, in practice one needs several formulas to describe the function.

Given two functions, f_1 with domain D_1 and f_2 with domain D_2, we define a third function f_3 with domain $D_1 \bigcup D_2$ as follows

$$f_3(x) = \begin{cases} f_1(x) & \text{if } x \text{ is in } D_1 \\ f_2(x) & \text{if } x \text{ is in } D_2. \end{cases}$$

Now to be a function, we must also require that for x in $D_1 \bigcap D_2$ that $f_1(x) = f_2(x)$. Otherwise our piecewise defined function will not be single valued. Luckily for us, however, most piecewise defined functions will have empty intersection.

2.2 More Examples

Example 2 Find the domain and graph the piecewise function

$$f(x) = \begin{cases} x+2 & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

The first thing to note here is that the intersection of the domains is empty. Thus, we are guaranteed to have a function with domain the union of the domains of its parts: $(-\infty,-1]\bigcup(-1,\infty) = \mathbb{R}$. Now, we just take the graph of the two parts of the function, and we stich them together at the point -1 on the x-axis. Also, technically speaking, we are only graphing a restricted function since graphing on the domain $= \mathbb{R}$ is impossible. Lastly, one should note that even through the two function agree on the point in which we sew them together, this is not required in general (this will however be required later when we begin discussing continous functions).

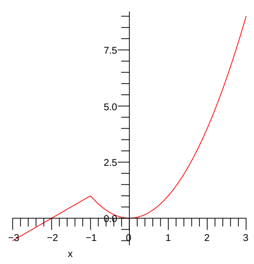

Figure 3: The graph of the above function from x = -3 to x= 3.

Example 3 Find the domain and graph the function

$$F(x) = \frac{3x + |x|}{x}$$

Again, the best we can hope to do for the domain is all of $\mathbb{R}$. However, here our domain is not allowed to contain the point $x = 0$. Thus, our domain is $(-\infty, 0) \bigcup (0, \infty)$. The graph of the function is:

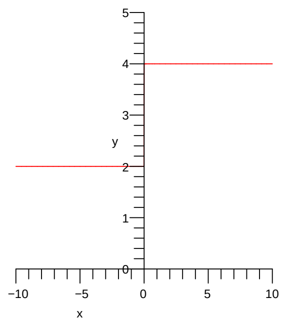

Figure 4: The graph of the above function from x = -10 to x= 10.

□

2.3 Chapter 1 Section 3

In this section we will see several ways of combing several functions to make new ones. This will facilitate our ability to graph several special kinds of functions.

2.3.1 Composing Functions

The most fundemental operation in mathematics is that of composition of functions. This is often called "fog" operation in high school. The operation takes two functions f and g and forms a new function $f \circ g$ with domain the domain of the function g. The only condition that the functions must satisfy is that the range of the function g is contained in the domain of the function f. Now we claim that $f \circ g$ is a function, so to define it, we must define its effect on every point of the domain.

$$\boxed{f \circ g(x) = f(g(x))}$$

Notice since we require that the range of g is contained in the domain of f this definition makes sense. The process goes as follows: the domain of g feeds the function g a value x. The function g puts out a value $g(x)$. The function f then takes the value $g(x)$ and outputs a value $f(g(x))$. If the notation seems to clumsy for you, you can replace $g(x)$ by x_2 and then we have $f(x_2)$.

Example 4 Given $f(x) = x^2 + 1$ and $g(x) = x + 1$ determine (the rule of) the function $f \circ g$ and $g \circ f$.

$$f \circ g(x) = \quad f(x+1) = \quad (x+1)^2 + 1 = \quad x^2 + 2x + 2 \tag{7}$$
$$g \circ f(x) = \quad g(x^2+1) = \quad (x^2+1) + 1 = \quad x^2 + 2. \tag{8}$$

□

Note that our example shows us a couple of things here. The first is that we do not have $f \circ g = g \circ f$ as we would expect from other binary operations that we are used to (such as addition and multiplication).

2.3.2 Algebraic Manipulations of Functions

Having discussed composoition of functions, we move onto discuss some algebraic manipulations that one can do with functions. These are the same manipulations that one can do with real numbers, namely add and multiply. So, given functions f and g, we wish to define fg and $f+g$. As with composition of functions, we need to define these new functions at every point.

$$\boxed{\begin{array}{c}(f+g)(x) := f(x) + g(x) \\ fg(x) := f(x)g(x)\end{array}}$$

This definition is the obvious definition. It's so obvious that it's hard to notice the subtlety of the definition. The left hand side is a statement, for example, about the function $f+g$ and it is to be evaluated at the point x. The right hand side actually says what this function does to that point x. Notice, that before discussing what it means to add fucntions, the right hand side already makes sense while the left hand side does not.

2.3.3 Specific Examples

Having defined the fundemental notions, we now define some specific examples:

1. Adding a constant function.
2. Mutiplying by a scalar multiple.
3. Shifting the Function.

Example 5

Consider the function $f(x) = x^2$. The graph of this function is the typical parable in the plane. Graph the functions

1. $f(x) + c$ for $c = 1, 2, 3$
2. $f(x + c)$ for $c = -2, -1, 1, 2$
3. $cf(x)$ for $c = -1, 1/2, 2, 5$

Now, the effect on the graphs in our example above hold in more general. In the sense that if the left hand graph depicts the graph of a random function then the right hand graph depicts the graph of a random function plus a scalar constant.

Notice, that at every point of the domain x the new function is the old function with shifted up by c units. Similiarly, we can shift the function over by c units:

Here the key is that the graph is shifted to the right by c units. Lastly, we examine the effect of multiplying by a constant.

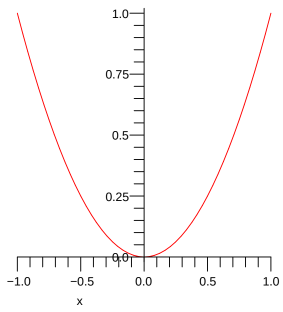

Figure 5: The graph of the above function $f(x) = x^2$ from x = -1 to x= 1

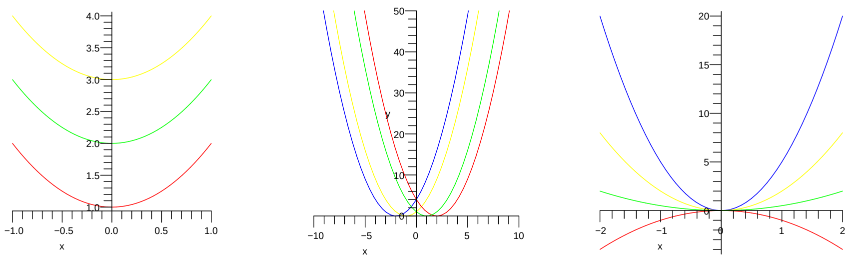

Figure 6: Parts 1 2 and 3

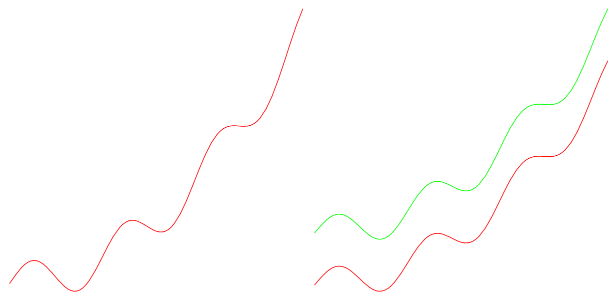

Figure 7: The graph of some function f and the graph of some function f depicted with the graph of some function $f + c$

Note that every point the original graph is stretched by a factor of c. These operations can also be combined to both move and stretch your graph.

Example 6 Sketch the graph of the equation $y = 3\,sin(x + 1/2) + 1/4$ by graphing a basic function and then using the rules above.

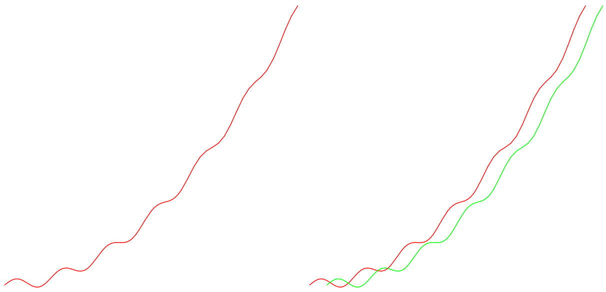

Figure 8: The graph of some function f and the graph of some function f depicted with the graph of some function $f \circ g$ where $g(x) = x - c$

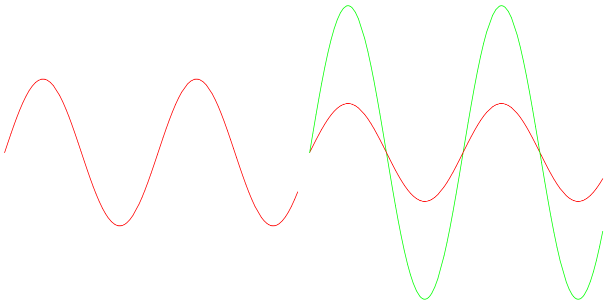

Figure 9: The graph of some function f and the graph of some function f depicted with the graph of some function cf

Let's start by breaking this function down to its basic parts. Obviously, the most familiar function in here is the sin function. Its graph should be familiar to you. We next consider $sin(x + 1/2)$. This is the sin function shifted over to the left by $1/2$. The next figure shows that

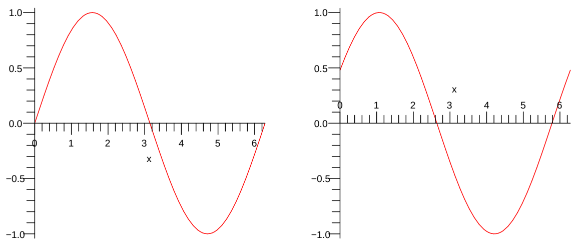

Figure 10: Graph of the sine function

We multiply by 3. Then lastly we add in the $1/4$.

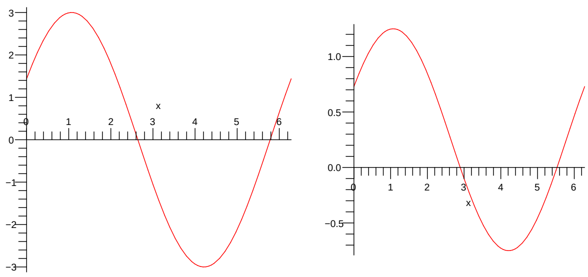

Figure 11: Graph of the sine function

3 Lecture IV: Section 1.6

In this lecture, we will see a concept that you might not be familiar with, *one to one*. The central topic in this lecture will be that of an inverse function. This is something that you should be familiar with. We will see that a necessary condition for a function to have an inverse is that the function be one-to-one.

3.1 Some definitions

Recall that a function has three parts:

1. A domain
2. A codomain
3. A rule.

Intuitively, an inverse function is a function that you would get by turning around the rule. Let's state the formal definition:

Definition: Given a function f a function g is said to be the **inverse function** of f if

1. For every x in the domain of g we have $f(g(x)) = x$.
2. For every y in the domain of f we have $g(f(x)) = y$.

We denote the function g by f^{-1} (note that $g^{-1} = f$).

It is important to realize that *not* every function has an inverse. For example, the function $f(x) = 5$ has domain $\mathbb{R}$ and is clearly a function. However, this function does not have an inverse. You can think of lack of an inverse function in terms of a T.V. show drama like C.S.I. Very often, they get bits of information such as the murdor victim was male. While every person is either female or male, there is no way to determine what person the body belongs to by simply noting their gender. On the other hand, if one of the investigators determine the dental records or the finger prints of the murder victim (or subject) it is possible to determine the identity of the unknown person. That is to say that the function from people to their genders is not invertible but the function from people to fingerprints is.

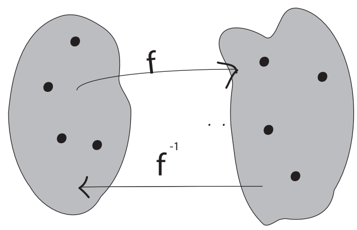

Figure 12: A function and its inverse

3.2 One to One Functions

The difference between the two functions discussed above is that the first function many different people have the same value (all males get assigned the same value in the codomain) but any given set of fingerprints belongs to only one person. This leads us to the formal definition of textitone to one.

Definition: A function f is said to be **one to one** if no two different values in the domain get assigned to the same value in the codomain. That is, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

Now, if we insist that a particular functin f is one-to-one with domain D and range R, then it makes sense to define $f^{-1}(y)$ as the unique value in the domain of f that gets mapped to y. Note, since the original function f is one to one, f^{-1} is actually a function! Summarizing:

Inverse Function Exists when original function f is one to one If f is one to one with domain D and range R then f^{-1} exists and has domain R and range D.

Example 2 Which of the following functins are 1 to 1. Which are invertible on their entire domain? If the function is not invertible on its entire domain, give a domain where it is.

1. $f(x) = sin(x)$
2. $f(x) = x^3$
3. $f(x) = 2^x$

From what I have above, these are really the same question.

1. Since $sin(0) = sin(\pi) = 0$ we have that the sin function is *not* one to one on all of $\mathbb{R}$. Thus it is not invertible on its domain. Notice, that we can restrict the domain to be (π, π) and the function is one to one there. Thus, the restricted function is invertible, but the inverse only has domain $(-\pi, \pi)$. This function is of course the arcsin function.
2. The function x^3 is one to one on its entire domain. This is because the function is increasing that is if $x_1 < x_2$ then $f(x_1) < f(x_2)$. The inverse of x^3 is $x^{(}1/3)$.

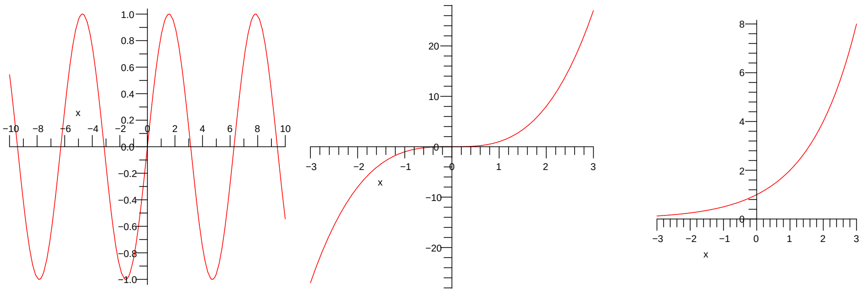

Figure 13: Graphs of the functions $\sin(x)$ x^3 and 2^x

3. The function 2^x is one to one throughout its domain (it is also an increasing function). The inverse function is $\log_2$. This function should be familiar to you. More generally, any exponential function, a^x is increasing for $a > 0$ and decreasing for $a < 0$, so it is one to one for all values of a not equal to 1. The inverse of a^x is written $\log_a$. Since the range of exponential functions are only the positive real numbers $(0, \infty)$, the domain of the logarithmic functions is only this same set $(0, \infty)$. That is to say something along the lines of $\ln_2(-5)$ is *not* defined.

When testing whether or not a given rule defined on a subset of $\mathbb{R}$ defines a function, we used the vertical line test. When testing whether or not a function has an inverse we will use a parallel arguement known as the horizontal line test.

The Horizontal Line Test A function f whose graph is given in the $xy - plane$ is one-to-one if and only if no horizontal line intersects its graph more than once.

Example 1 Show all the functions below fail the vertical line test and find explicit values of the function that get mapped to the same point of the range.

1. $f(x) = x^2$
2. $g(x) = x + x^4$
3. $h(x) = cos(x)$

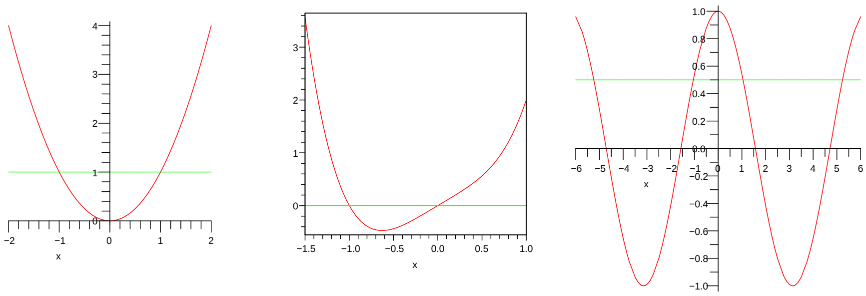

Figure 14: Graphs of the functions $\sin(x)$, x^3, and 2^x

1. $f(-1) = f(1)$ as a matter of fact since f is even $f(-a) = f(a)$ for every a. This is why it is necessary to make a choice when defining the square root function.
2. In the graph above it appears as if though $f(x) = 0$ for two seperate values of x. We find what values of x the function vanishes by the usual algebraic methods:

$$\begin{aligned} g(x) =& 0 && (9)\\ &\Rightarrow x^4 + x = 0 && (10)\\ &\Rightarrow x\,(x^3+1) = 0 && (11)\\ &\Rightarrow x = 0 \text{ or } x^3+1 = 0 && (12)\\ &\Rightarrow x = 0 \text{ or } x^3 = -1 && (13)\\ &\Rightarrow x = 0 \text{ or } x = -1 && (14)\\ & && (15) \end{aligned}$$

   Let's double check that we are right: $g(-1) = (-1)^4 + 1 = -1 + 1 = 0$ and $f(0) = 0^4 + 0 = 0$.
3. For our first two examples, we got exaclty two values getting sent from the domain to a single value in the range. However, for the cosine function, we have infinite number of values getting sent to any given value in the range. For example, $1/2 = \cos(\pi/3) = \cos(\pi/3 + 2\,pi) = \cos(\pi/3 + 4\,pi) = \ldots \cos(\pi/3 + 2\,n\pi)$ for every integer n (note that this can't possibly happen with a polynomail since after factoring we get only finite number of places where the polynomail can equal zero and by shifting any other point).

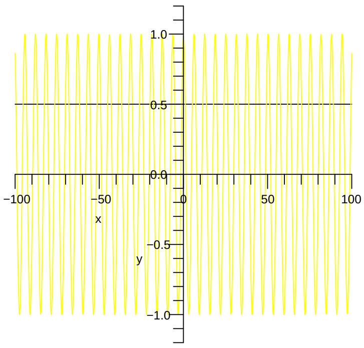

Figure 15: The graph of the $\cos(x)$ over the domain $[-100, 100]$ plotted along with the vertical line determined by the equation $y = 1/2$ (in black)

3.3 Examples of Finding the Inverse of a Given Function

In this section, we give some concrete methods for finding the inverse of a given polynomial function. While the method doesn't always work, it's the best that we can do!

Find the Inverse of a One-to-One function

1. Write the function f in the form $y = f(x)$.
2. Solve the equation for x in terms of y.
3. To express f^{-1} as a function of x simply replace y by x in the resulting equation.

Note that the third step is purly formal, and it is done to keep with the tradition of whenever an equation is given which determines a function to have x written for the independent variable and y given for the dependent variable.
Example 3 Using the procedure above, find an expression for the inverse of the function $f(x) = 2\,x^3 + 3$ (this is exercise 26 on page 75 of your text).

This is straight foward:

$$\begin{aligned}
\textbf{Step 1 } & y = 2\,x^3 + 3. && (16)\\
\textbf{Step 2 } & y = 2\,x^3 + 3 && (17)\\
& \Rightarrow y - 3 = 2\,x^3 && (18)\\
& \Rightarrow (y-3)/2 = x^3 && (19)\\
& \Rightarrow \sqrt[3]{\frac{y-3}{2}} = x && (20)\\
\textbf{Step 3 } & f^{-1}(x) = \sqrt[3]{\frac{x-3}{2}} && (21)
\end{aligned}$$

□

3.4 Logs

In this section, we focus on the log function which is a particular example of an inverse function mentioned in example 1. To state what a log is again, we give a formal definition.
Definition: $\log_a(x)$ is definined to be the unique number b so that $a^b = x$.
The following are some of the basic properties of log functions:

1. $\log_a(x\,y) = \log_a(x) + \log_a(y)$
2. $\log_a(x^b) = b\,log_a(x)$

Since a log function in an inverse function, it is uniquely determined by its inverse. Thus all of the above properties can be derived from the corresponding properties of exponential functions. For example:

$$\begin{aligned}
\log_a(xy) &= \log_a(a^{\log_a(x)} a^{\log_a(y)}) && (22)\\
&= \log_a(a^{\log_a(x)+\log_a(y)}) && (23)\\
&= \log_a(x) + \log_a(y) && (24)
\end{aligned}$$

In homework, you will be asked to solve a few equations using logs.

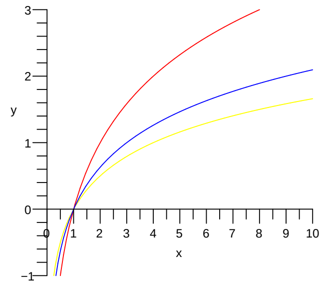

Figure 16: $\log_4(x)$ in yellow, $\log_3(x)$ in blue, and $\log_2(x)$ in red

Example 4 Solve $3^{x+6} = 7$ for x.

$$3^{x+6} = 7 \tag{25}$$
$$\Rightarrow x + 6 = log_3(7) \tag{26}$$
$$\Rightarrow x = log_3(7) - 6 \approx -4.228756251. \tag{27}$$

4 Lecture V: Section 2.2 and 2.3

4.1 Chaper II Section 2: The Limit of a Function

Limits are the nuts and bolts of calculus. This is the concept that will allow us to define the concepts of derivatives and integrals later in the course (the concept of the limit will the be subsequently hidden under the machinary). The concept of a limit was first used by the founding fathers of calculus such as Newton and Leibnez. However, they did not define "the limit" precisely. Newton first published his work displaying the fundemental theorem of calculus in 1666. However, the desire to formulize these ideas wasn't presented until 1734. A perscise definition wasn't given until 1816 by Bernhard Bolzano (and more publicly by Augustin Louis Cauchy in 1819). This is a 150 year window!! We will take the approach that if it was good enough for Newton it will be good enough for us. While this approach has disadvantages, the formal definition of a limit would be without a doubt the hardest single concept would cover in this class. We will therefore be using an intuitive notiion of what a limit is.

Suppose that you were going to your favorite ice creme place in your home town. However, in the time that you have been in college, they tore it town and put up a Starbucks. When you got to the Starbucks, you call your Mom, and she asks "where were you going"? The answer of course would be to the ice creme place. In math, the limit is a similiar concept.

4.1.1 Definitions and Basic Idea

Example 1 Consider the function

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 1 \\ 10 & \text{if } x = 1 \end{cases} \tag{28}$$

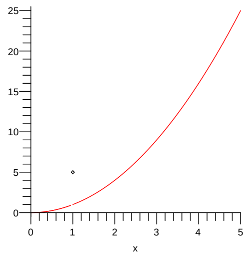

Figure 17: Graph of $f(x)$

The informal question we could ask a traveler walking up the curve of the graph heading for $x = 1$ is "where are you going." His answer would probably be "the point (1,1)." Now, he wouldn't know that that point is not a point on the graph until he got there. As a matter of fact, he could come arbitrarly close to it, for example:

x	f(x)
0	0
0.5	0.25
0.75	0.5625
0.90	0.81
0.95	0.9025
0.99	0.9801
0.999	.998001

Table 1: Several Points on the graph of the function as x gets closer to 1 from the negative side

Fusturated, the traveler says, I knew the point (1,1) was here before. However, now the point (1,10) is in this graph. Every indication says that its here, so he tries to come from a different place. This time, he comes from the positivie side.

Again, coming from this side, the traveler would conclude that he is heading for the point (1,1). In this example, we write

$$\lim_{x \to 1} f(x) = 1 \tag{29}$$

x	f(x)
2	4
1.5	2.25
1.25	1.5625
1.05	1.1025
1.01	1.0201
1.001	1.002001
1.0001	1.00020001

Table 2: Several Points on the graph of the function as x gets closer to 1 from the positive side

Notice, we are saying this because the function comes arbitrarily close to this point even though the actual value of the function is $f(x) = 10$. We will later define a function to be continuous when such things don't happen. That is when the limit and the actual value agree.

□

"Definition" 1 We write

$$\lim_{x \to a} f(x) = L \tag{30}$$

and say that "the limit of $f(x)$, as x approaches a equals L" if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a (ON BOTH SIDES OF a) but not equal to a (I use the quotes because this is not a formal definition, what does it mean to be "arb" close ... for a formal definition see section 2.4 in the text).

Example 2 Find

$$\lim_{x \to 2} x^2 + x - 2. \tag{31}$$

Now, unlike the previous example, this graph doesn't have any "holes". Thus, we expect the limit of the function to actually be the value of the function.
That is

$$\lim_{x \to 2} x^2 + x - 2 = f(2) = 2^2 + 2 - 2 = 4. \tag{32}$$

□

4.1.2 Noexistance of Limits

Now, this concept so far might seem easy. But, there is always a monkey wrench. Like when we discussed inverses in the last class, we were able to give a nice definition of an inverse function, but unfortunetely, the math gods do not care how nice your definition is, because very often the inverse did not exist. Things

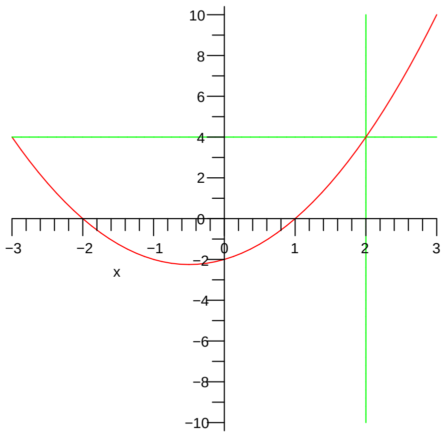

Figure 18: Here $f(x) = 4$ and so does the limit!

here look equally as bleak. A first obvious way that the limit may fail to exist is by simply looking at our "definition". In the definition, we require that we get the same value coming from "both sides". So, we can easily concock an example where this doesn't happen, and so the limit would not exist.

Example 3 Discuss

$$\lim_{x \to 7} f(x) \tag{33}$$

for

$$f(x) = \begin{cases} 5 & \text{if } x < 7 \\ 10 & \text{if} \geq 7 \end{cases} \tag{34}$$

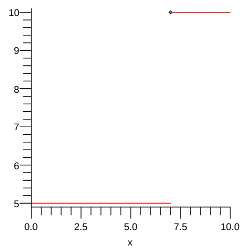

Figure 19: Graph of $f(x)$

Notice as we approach $x = 7$ from the left side we expect the value to be 5 but as we approach from the right side we expect our value to be 10. This means (according to our definition) that our limit does not exist.

□

Even worse, sometimes our function doesn't approach any value. This is the case with the "big dig". Will it be done next week? Next year? Next decade? Anybody who would conjecture a finish date would be a fool. Similiary, many functions have this property.

Example 4 Discuss

$$\lim_{x \to 1} 1/(x-1)^2 \tag{35}$$

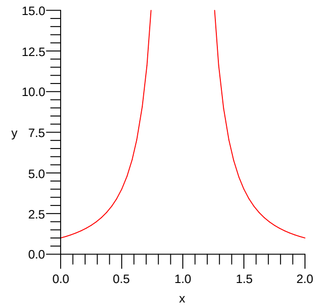

Figure 20: Graph of $f(x)$

Let's begin by putting some values into a table like we did with our first example.

x	f(x)
0.00	1.0000
0.50	4.0000
0.80	25.000
0.90	100.00
0.95	400.00
0.99	10,000

Table 3: Several Points on the graph of the function as x gets closer to 1 from the negative side

A similiar thing happens when approaching from the right hand side. Now, since the function does not come close to any real number our limit does not exist. However, mathematicians like to be a little more perscise than this. The reason for the fact that the limit not existing is decididly different from the reason the limit did not exist in the previous example. In this exampe, the function gets larger and larger. For this reason, we use the notation:

$$\lim_{x \to 1} f(x) = \infty. \tag{36}$$

and say "the limit of $f(x)$, as x approaches a, is infinity".

Note that this is only a notation! The limit **DOES NOT EXIST**! This is one the largest errors calculus students make. The limit can not possibly exist since **infinity is not a NUMBER**!! However, we still use the notation above no matter how misleading.

$\square$

4.1.3 One sided limits

In our first example of a non-existant limit (example 3 in the previous section), we ran into difficulty because we got two different answers coming from two different sides. However, we would still like to say something about the nature of the function when coming from the two different sides.

"Definition" 2 We write

$$\lim_{x\to a^-} f(x) = L_1 \tag{37}$$

$$\lim_{x\to a^+} f(x) = L_2 \tag{38}$$

to mean that as the function f comes arbitrarily close to L_1 when x comes close to a coming from the negative sides (positive side resp.). Note that in example 3 in the previous section, we have that $\lim_{x\to 7^-} f(x) = 5$ and $\lim_{x\to a^+} f(x) = 10$.

Some notes:

1. The limit exists if and only if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist and $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$ (they are equal).
2. There exist functions whose one sided limit does not exist from either side and whose limit does not go off to infinty, namely random functions.

4.1.4 Vertical Asymptote

The Last topic that needs to be briefly touched upon in this section is that of asymptote.

Definition 3 The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if any of the following statements is true:

$$\lim_{x\to a^-} f(x) = \infty \tag{39}$$

$$\lim_{x\to a^+} f(x) = \infty \tag{40}$$

$$\lim_{x\to a^-} f(x) = -\infty \tag{41}$$

$$\lim_{x\to a^+} f(x) = -\infty \tag{42}$$

where the above are defined in the obvious way based on example 4. We can think of an asymptote as a vertical line in which the functions graph can approach but never reach.

Example 3 Find the vertical asymptotes of the functions

$$f(x) = \frac{2\,x}{x-3} \tag{43}$$

We notice as x gets close to 3, our denomenator gets close 0. In the mean time the numerator gets close to 6. This shows us that as x gets to 3, our function $f(x)$ is going off towards ∞. More percisely, coming from the left $x-3$ is negative, so the function goes to -∞ and coming from the right $x-3$ is positive, so the function goes to ∞.

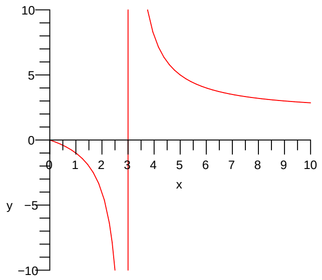

Figure 21: Graph of $f(x)$

We notice in the graph above that the function comes closer and closer to the vertical line $x = 3$ from both sides, but it never touches it.

Answer The vertical asymptote is the line $x = 3$. □

4.2 Limit Laws

This section of our book we finally get to see and use some techniques that will be extremly useful throughout you calculus studies. We will be presented with some real life limits, and we will be given a tool book of techniques that will be used in hopes of figuring out what the limit is.

4.2.1 The Laws

There are two main limit laws to consider (the book presents 11 but, the other 9 are obvious consequences of the following 2):

Limit Laws Suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist then:

$$\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \qquad \textit{Sum Law} \tag{44}$$

$$\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a}(f(x) \lim_{x\to a} g(x) \qquad \textit{Product Law} \tag{45}$$

The sum law implies

$$\lim_{x\to a}(f(x) - g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) \qquad \textit{Subtraction} \tag{46}$$

and the product gives

$$\lim_{x\to a}(c\, f(x)) = c \lim_{x\to a} f(x) \qquad \textit{Scalar Multiple Law} \tag{47}$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \qquad \textit{Quotient Law} \tag{48}$$

$$\lim_{x\to a} f(x)^n = (\lim_{x\to a} f(x))^n \qquad \textit{Power Law} \tag{49}$$

Whenever c is a constant and $\lim_{x\to a} g(x) \neq 0$ in the quotioent rule statement.

These rules along with the following assumption:

$$\lim_{x\to a} f(x) = f(a) \tag{50}$$

for any rational function with a in its domain (a rational funtion is any function that can be written as the quotient of two polynomials).

4.2.2 Some Examples and Calculations

(Not written out here)

4.2.3 The Squeeze Thm.

So far, besides polynomials and rational functions, we can not solve a tremendous amount of limit problems. We do not know how to solve problems involving $\sin, \cos, \ln$, or exponetials. We do however, get a little bit of help from the so called squeeze theorem.

The Squeeze Theorem Given functions f, g, and h with $f(x) \leq g(x) \leq h(x)$ for all x, if we have that

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = c \text{ then} \tag{51}$$

$$\lim_{x\to a} g(x) = a. \tag{52}$$

While this is a more complicated statement then others we have seen so far in the course, it is however the only way we can solve more complicated limit problems such as $\lim_{x\to 0} \sin(x)/x = 1$ which is necessary to know when trying to figure out the derivative on the sin function.

Example n Show that

$$\lim_{x\to 0} x^2\, e^{(\cos(\frac{\pi}{x})} = 0 \tag{53}$$

$$\tag{54}$$

Since the largest $\cos(x)$ can ever be is 1 we have that our function is bounded above by $x^2\, e$. Similiarly, the smallest $\cos(x)$ can ever be is -1. Thus,our function is bounded below by $x^2\, e^{-1}$. Moreover $\lim_{x\to 0} x^2\, e^{-1} = \lim_{x\to 0} x^2\, e^{=}0$. Thus by the statement of the squeeze theorem we have that $\lim_{x\to 0} x^2\, e^{(\cos(\frac{\pi}{x})} = 0$

□

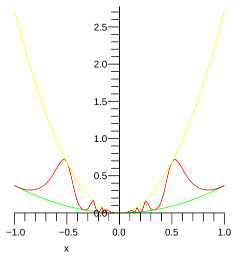

Figure 22: Graph of $f(x)$

5 Lecture VIII: Section 2.7

5.1 The falling ball

When Newton first discussed derivatives, he always took derivatives with respect to time. I will give an example based on the simple physical phenomena of a rising and falling ball. Plotted with respect to time, the plot of the ball position is:

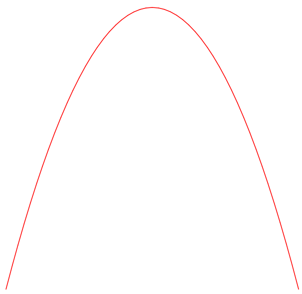

Figure 23: Graph of balls motion with respect to time

(It's important to notice that this graph is a graph with respect to time and not with respect to position!) Intuitivly, the ball "stops for a moment" at its highest point. By this I mean, the velocity is zero. Moreover, by the intermediate value theorem, this statement has to be true (since the ball at one moment is moving up and at the next its moving down). However, the ball never actually "stops". This means, that for any change in time, there is a change is position. That is:

$$\frac{\Delta x}{\Delta t} \neq 0. \tag{55}$$

To make the problem something we can get our hands on, we put a coordinate system on the parable. We can assume that the highest the ball reaches is one unit in one unit of time. This means that our function is:

$$f(t) = 4\,t\,(1-t) \tag{56}$$

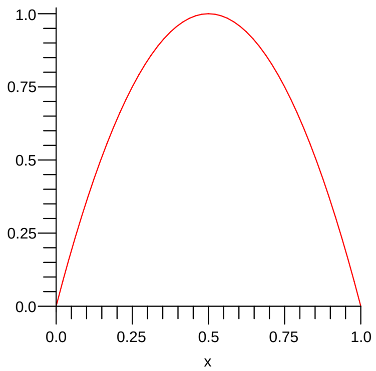

Figure 24: Graph of balls motion with respect to time with a coordinate system in place

Now everybody who has had elementary school physics knows that velocity is the change in distance of time. Let's measure some values of the velocity as the ball gets closer and closer to its zenith (which occurs at t=1/2). At t=.4 (=2/5) height of the ball is $f(.4) = 4(.4)(1-.4) = .96$.And in the next 1/10 of a second, the ball climbs the additional 0.04 units up. That is we have:

$$v_{ave} = \frac{\Delta x}{\Delta t} = \frac{0.04}{0.1} = 0.4\frac{\text{(height units)}}{\text{(time units)}}. \tag{57}$$

What does this calculation represent geometrically? To this end, we go off on what might seem a tangent (hehe) for just a moment. Remember the second appendix that you guys did homework for. In that section, you reviewed how to make come up with an equation for a (straight line) line simply with the aid of the knowledge of two points on the line (x_1, y_1) and (x_2, y_2). The equation used was $y = m\,x + b$. This should be an equation everybody should be familiar with from your early days in highschool. The m was the slope of the line and was defined as:

$$\text{slope of line} = m := \frac{y_2 - y_1}{x_2 - x_1}. \tag{58}$$

and the b is the so-called y-intersept. The place where the line meets the y-axis. This b also only depends on two points as well, namely:

$$b = \frac{y_1\,x_2 - y_2\,x_1}{x_2 - x_1} \tag{59}$$

The value of b right now is now that important, what is important is that it is that it only depends on the two points in the plane. Going back to our veloctiy example, we saw that our average velocity over the interval in question is was found by the exact same value as the slope of the tangent line was found by. That is:

$$v_{ave} = \begin{cases} \text{Slope of unique line through} \\ \text{the points } (.4,\ .96) \text{ and } (.5,\ 1.0) \end{cases} \tag{60}$$

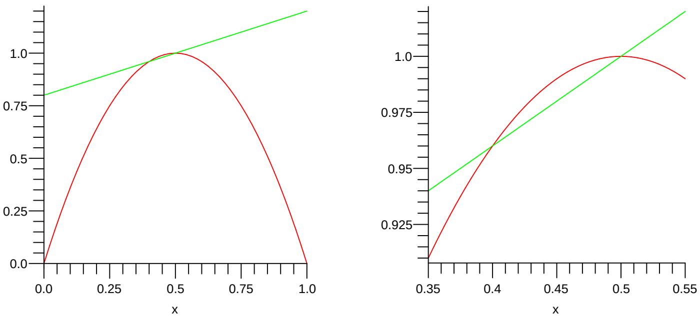

Figure 25: Graph of the function along with the unique line that passes through the points in question. The graph on the left is plotted on the domain [0,1] and the right is plotted on the domain[0.35, 0.55]

What happens to the tangent line as we take our x_1 value closer and closer to the point where we expect the velocity to become zero, namely $x = 1/2$? By the equivalence shown above, we can do this geometrically by simply drawing the unique line through the two points $x_2 = 1/2$ and $x_1 = 1/2 - \delta$where δ becomes smaller and smaller (the so called **secant lines**)?

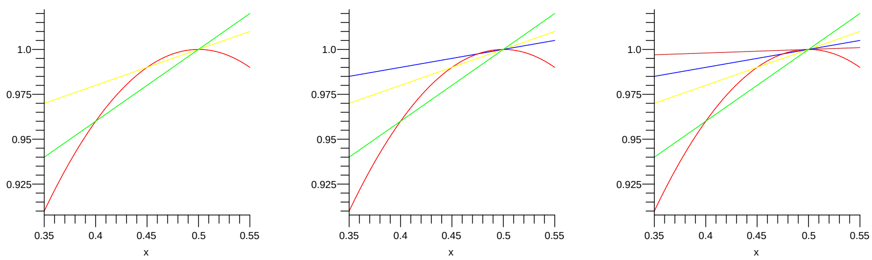

Figure 26: Unique lines through the points (1/2,1) and $(x_1,\ f(x_1))$ as x_1 approaches the point $x = 1/2$.

We can clearly see that the slope of the tangent line approaches zero as x_1 approaches $1/2$. Statements like this should be familiar to us from the previous sections on limits. We are saying that:

$$\lim_{x_1 \to 1/2} (\text{Slope of unique line through these points}) = 0. \tag{61}$$

.
And from our discussion above, this is saying that:

$$\lim_{x_1 \to 1/2} \frac{f(1/2) - f(x_1)}{1/2 - x_1} = 0. \tag{62}$$

And of course this is what we *expected* to be the velocity of the ball at the top of its climb.

5.2 Moving Away From the Example

We now write the above in a more general form, and we make this a definition.
Definition 1 *The slope of the Tangent Line* at the point a is defined to be

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}. \tag{63}$$

In the next section, we will call this limit the derivative of the function f as the point a. That is, we will define the slope of the tangent line at a to be the derivative at a. Derivatives make up the first half of calculus, and we will be studying their properties for the next several months. They are one of the most useful objects and math and the scope of their applications are nearly unlimitied. Their introduction is one of the most prominent signs that a new era of mathematics had begun. While the formula above may look rather complicated and tedious to evaluate, we will develop a mechanical process for evaluating derivatives that will the evaluation of nearly every function very routine. However, for the next couple of sections, we have to "pretend" that we do not know any of this hard core machinary and grind out derivatives (I mean slopes of tangents lines) by the definition.
Example 1 Use the definition to show that the slope of the tangent line of the function in the previous

section at $x = 1/2$ is indeed zero.

$$\lim_{x\to 1/2} \frac{f(x) - f(1/2)}{x - 1/2} = \lim_{x\to 1/2} \frac{4\,x\,(1-x) - (4)\,(1/2)\,(1-1/2)}{x - 1/2}. \tag{64}$$
$$= \lim_{x\to 1/2} \frac{4x - 4x^2 - 1}{x - 1/2} \tag{65}$$
$$= \lim_{x\to 1/2} \frac{4x^2 - 4x + 1}{1/2 - x} \tag{66}$$
$$= \lim_{x\to 1/2} \frac{x^2 - x + 1/4}{1/4(1/2 - x)} \tag{67}$$
$$= \lim_{x\to 1/2} \frac{(x - 1/2)(x - 1/2)}{1/4(1/2 - x)} \tag{68}$$
$$= \lim_{x\to 1/2} \frac{(x - 1/2)(x - 1/2)}{-1/4(x - 1/2)} \tag{69}$$
$$= \lim_{x\to 1/2} \frac{(x - 1/2)}{-1/4} \tag{70}$$
$$= 0. \tag{71}$$

□

5.3 Examples

Believe it or not, these problems are easier when we do not evaluate at a specific point (like we did above at the point $a = 1/2$). Let's start with an easy example.
Example 2 Find the slope of the tangent line of the function:

$$f(x) = x^2. \tag{72}$$

at the point $x = a$.

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{x^2 - a^2}{x - a} \qquad \text{Definition} \tag{73}$$
$$= \lim_{x\to a} \frac{(x-a)(x+a)}{x - a} \qquad \text{Factor} \tag{74}$$
$$= \lim_{x\to a} x + a \qquad \text{Cancel } x - a \text{ terms} \tag{75}$$
$$= 2\,a \qquad \text{Evaluate} \tag{76}$$

□

Let's try a slightly harder example and find the slope at an arbitrary point a.
Example 3 Find the slope of the tangent line of the function:

$$f(x) = \frac{4}{x + 3}. \tag{77}$$

at the point $x = a$.

$$\begin{aligned}
\lim_{x\to a}\frac{f(x)-f(a)}{x-a} &= \lim_{x\to a}\frac{(4/(x+3)-4/(a+3))}{x-a} && \text{Definition} && (78)\\
&= \lim_{x\to a}\frac{4(a+3)/(x+3)(a+3)-4(x+3)/(a+3)(x+3)}{x-a} && \text{Common den.} && (79)\\
&= \lim_{x\to a}\frac{(4(a+3)-4(x+3))/(x+3)(a+3)}{x-a} && \text{Subtract fraction} && (80)\\
&= \lim_{x\to a}\frac{4(a+3)-4(x+3)}{(x-a)(x+3)(a+3)} && \text{Group denomenators} && (81)\\
&= \lim_{x\to a}\frac{4(a-x)}{(x-a)(x+3)(a+3)} && \text{Cancel 12's} && (82)\\
&= \lim_{x\to a}\frac{-4}{(x+3)(a+3)} && \text{Cancel } a-x \text{ terms} && (83)\\
&= \frac{-4}{(a+3)^2} && \text{Rat. fun. continuous} && (84)
\end{aligned}$$

□

If you have had calculus before, you may recognize the way above as the derivative of $f(x) = \frac{4}{x+3}$ evaluated at the point $x = a$. If you haven't had calculus before, please dismiss this line from the testomony of the instructor in front of you.

As with the previous homework, the square roots are slightly trickier. And like the last homework, "un"rationalizing will come to our rescue.

Example 4 Find the slope of the tangent line of the function:

$$f(x) = \sqrt{x}. \tag{85}$$

at the point $x = a$.

$$\begin{aligned}
\lim_{x\to a}\frac{f(x)-f(a)}{x-a} &= \lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{x-a} && \text{Definition} && (86)\\
&= \lim_{x\to a}\frac{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})}{(x-a)(\sqrt{x}+\sqrt{a})} && \text{Unrationalize} && (87)\\
&= \lim_{x\to a}\frac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})} && \text{Simplify} && (88)\\
&= \lim_{x\to a}\frac{1}{\sqrt{x}+\sqrt{a}} && \text{Cancel } (x-a) \text{ terms} && (89)\\
&= \frac{1}{2\sqrt{a}} && \text{Evaluate} && (90)
\end{aligned}$$

□

Here's another similiar problem with just some extra junk floating around.

Example 5 Find the slope of the tangent line of the function:

$$f(x) = \sqrt{3\,x+2}. \tag{91}$$

at the point $x = a$.

$$\begin{aligned}
\lim_{x\to a} \frac{f(x)-f(a)}{x-a} &= \lim_{x\to a} \frac{\sqrt{3\,x+2}-\sqrt{3\,a+2}}{x-a} && \text{Definition} \quad (92)\\
&= \lim_{x\to a} \frac{(\sqrt{3\,x+2}-\sqrt{3\,x+2})(\sqrt{3\,x+2}+\sqrt{3\,a+2})}{(x-a)(\sqrt{3x+2}+\sqrt{3a+2})} && \text{Unrationalize} \quad (93)\\
&= \lim_{x\to a} \frac{3(x-a)}{3(x-a)(\sqrt{3x+2}+\sqrt{3a+2})} && \text{Simplify} \quad (94)\\
&= \lim_{x\to a} \frac{1}{\sqrt{3\,x+2}+\sqrt{3\,a+2}} && \text{Cancel } 3(x-a) \text{ terms} \quad (95)\\
&= \frac{1}{2\sqrt{3\,a+2}} && \text{Evaluate} \quad (96)
\end{aligned}$$

□

6 Lecture X: Section 3.1

Today, we are ready to start to use and justify some of the familiar properties of derivatives. The main properties one has to be able to use are: (for any two functions f and g and constants a and b).

$$\begin{aligned}
\frac{d}{dx}\left(a\,f\left(x\right)+b\,g\left(x\right)\right) &= a\,\frac{d}{dx}f\left(x\right)+b\,\frac{d}{dx}g\left(x\right) && \text{Linearity} \quad (97)\\
\frac{d}{dx}\left(f\left(x\right)g\left(x\right)\right) &= \left(\frac{d}{dx}f\left(x\right)\right)g\left(x\right)+f\left(x\right)\frac{d}{dx}g\left(x\right) && \text{Product Rule} \quad (98)\\
\frac{d}{dx}(f\circ u(x)) &= \frac{d}{du}f(u(x))\frac{d}{dx}u(x) && \text{Chain Rule} \quad (99)
\end{aligned}$$

These three properties again correspond to the three main operations that we covered in the first week: addition, multiplication and composition. In today's lecture, we will cover the first of these three properties, as well as some special cases of the other two.

6.1 Derivatives of Constants and x^n

The easiest general derivative to take is that of a constant.
The question is essentially "if I spent all day at one point how fast would I be going". The obvious answer to this is 0 miles per hour. That is to say that the derivative of a constant function is zero. To give a formal justification:

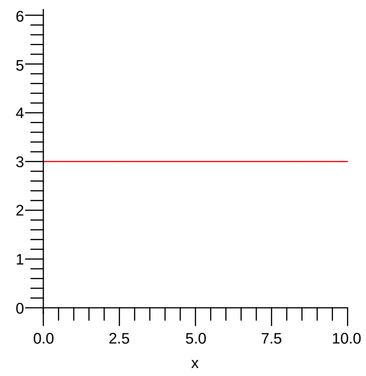

Figure 27: Graph of a Constant Function

$$\frac{d}{dx}(constant) = \lim_{\Delta x \to 0} \frac{constant(x + \Delta x) - constant(x)}{\Delta x} \tag{100}$$
$$= \lim_{\Delta x \to 0} \frac{constant - constant}{\Delta x} \tag{101}$$
$$= \lim_{\Delta x \to 0} \frac{0}{\Delta x} \tag{102}$$
$$= 0. \tag{103}$$

This shows without a shadow of a doubt that the derivative of any constant function is zero.

The next most basic class of functions are the functions of the form $f(x) = x^n$ for real numbers n (note this includes functions of the form $f(x) = \sqrt{x}$ in addition to functions of the form $f(x) = x^2$).

The (General) Power Rule For every real number n, we have $\frac{d}{dx}x^n = n\,x^{n-1}$

Before going into why this is true, let's look at a few examples to clarify what exactly this is saying.

Example 1 Find the derivative of the functions $f(x) =$:

$$1)x^3 \qquad 2)\sqrt{x} \qquad 3)x^{7/2} \qquad 4)\frac{1}{x^3}. \tag{104}$$

$$1)\ \frac{d}{dx}x^3 = 3\,x^2 \qquad \text{Here } n = 3 \tag{105}$$

$$2)\ \frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}\,\frac{1}{x^{1/2}} = \frac{1}{2\sqrt{x}} \qquad \text{Here } n = 1/2 \tag{106}$$

$$3)\ \frac{d}{dx}x^{7/2} = \frac{7}{2}\,\frac{1}{x^{7/2-1}} = \frac{7}{2\,x^{5/2}} \qquad \text{Here } n = 7/2 \tag{107}$$

$$4)\ \frac{d}{dx}\frac{1}{x^3} = \frac{d}{dx}x^{-3} = -3\,x^{-3-1} = -3\,x^{-4} = -\frac{3}{x^4} \qquad \text{Here } n = \text{-3} \tag{108}$$

□

These problems are extremly easy to do, much easier than using the definition. Suprisingly, the most common type of error in these sorts of errors is simple arithmatic errors. For example, people often take -3-1 = -2 or something similiar. The general power rule above is a consequence of the product and chain rules in general. However, when we restrict ourselves to positive integers, we can prove a weaker form of the theorem (the book calls it in this form as the power rule with the word *general* excluded). It is an amazing fact that the slope of function comes out to such a simple form. Often, geometrical constructions (for example, finding the slope of a general curve at every point) have results that are quite exotic. The ultimate example being the number π.

Validification of the power rule for non negative integers n.

$$\begin{aligned}
\frac{d}{dx}x^n &= \lim_{\Delta x\to 0}\frac{(x+\Delta x)^n - x^n}{\Delta x} && (109)\\
&= \lim_{\Delta x\to 0}\frac{(x^n + n\,x^{n-1}\,\Delta x + \cdots n\,x\,\Delta x^{n-1} + \Delta x^n) - x^n}{\Delta x} && (110)\\
&= \lim_{\Delta x\to 0}\frac{(n\,x^{n-1}\,\Delta x + \cdots n\,x\,\Delta x^{n-1} + \Delta x^n)}{\Delta x} && (111)\\
&= \lim_{\Delta x\to 0}(n\,x^{n-1} \ + \cdots n\,x\,\Delta x^{n-2} + \Delta x^{n-1}) && (112)\\
&= n\,x^{n-1}. && (113)
\end{aligned}$$

□

Again, it's important to realize this does not say anything about the functions $f(x) = \sqrt{x}$ or any function where n is not a nonnegative integer.

6.2 Constant Multiple and Sum Rule

The first rule that I expressed at the beginning of this section can less precisely expressed in terms of two seperate rules.

1. $\frac{d}{dx}(c\,f(x)) = c\frac{d}{dx}f(x)$ Constant Multiple Rule
2. $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x))$ Sum Rule

The justification for these two laws again follows right from the definition of the derivative. We first let $g(x) = c\,f(x)$ for some constant c.

$$
\begin{aligned}
\frac{d}{dx}\,(cf(x)) &= \frac{d}{dx}g(x) && \text{Definition of } g(x) && (114)\\
&= \lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x} && \text{Definition of Derivitive} && (115)\\
&= \lim_{\Delta x\to 0}\frac{cf(x+\Delta x)-\,cf(x)}{\Delta x} && \text{Definition of } g(x) && (116)\\
&= \lim_{\Delta x\to 0} c\left(\frac{f(x+\Delta x)-f(x)}{\Delta x}\right) && \text{Basic Distributivity Property} && (117)\\
&= c\left(\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\right) && \text{Constant Multiple Rule For Limits} && (118)\\
&= c\,\frac{d}{dx}f(x) && \text{Definition.} && (119)
\end{aligned}
$$

□

Not much harder is the sum law. Here, we let $g(x) = f_1(x) + f_2(x)$.

$$
\begin{aligned}
\frac{d}{dx}\,(f_1(x)+f_2(x)) &= \frac{d}{dx}g(x) && \text{Definition of } g(x) && (120)\\
&= \lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x} && \text{Definition of Derivitive} && (121)\\
&= \lim_{\Delta x\to 0}\frac{(f_1+f_2)\,(x+\Delta x)-(f_1+f_2)\,(x)}{\Delta x} && g(x)\text{ Def} && (122)\\
&= \lim_{\Delta x\to 0}\frac{f_1(x+\Delta x)+f_2(x+\Delta x)-(f_1(x)+f_2(x))}{\Delta x} && \text{Def. of Sum} && (123)\\
&= \lim_{\Delta x\to 0}\frac{f_1(x+\Delta x)-f_1(x)+f_2(x+\Delta x)-f_2(x)}{\Delta x} && \text{Rearrange Terms} && (124)\\
&= \lim_{\Delta x\to 0}\frac{f_1(x+\Delta x)-f_1(x)}{\Delta x}+\frac{f_2(x+\Delta x)-f_2(x)}{\Delta x} && \text{Distribute} && (125)\\
&= \frac{d}{dx}f_1(x)+\frac{d}{dx}f_2(x) && \text{Definition} && (126)
\end{aligned}
$$

□

Using the properties developed thus far in this section, we now have the ability to take the derivative of any polynomial.

Example 2 Find the derivative of the function $f(x) = x^3 + x$:

$$
\begin{aligned}
\frac{d}{dx}\left(x^3+x\right) &= \frac{d}{dx}\left(x^3\right)+\frac{d}{dx}(x) && \text{Sum Property} && (127)\\
&= 3\,x^2+1 && \text{Two Applications of Power Rule} && (128)
\end{aligned}
$$

□

6.3 Derivatives of Exponential Functions

The easiest function to take the derivative of is the function e^x.

$$\frac{d}{dx}e^x = e^x. \tag{129}$$