Afte THE PRINCIPLE OF GENERAL COVARIANCE : REPRISE

After this quite long mathematical detour, where we learned how to describe mathematically the geometry of curved spacetime, we ~~are~~ ready to describe the theory of gravitation which follows from Einstein's equivalence principle.

To this aim, we must be able to explain:

→ HOW ~~THE GRAVITATIONAL FIELD~~ GRAVITATION INFLUENCES THE BEHAVIOUR OF MATTER (1)

→ HOW THE MATTER [~~CURVES SPACETIME~~] ~~DETERMINING~~ DETERMINES ~~THE GRAVITATIONAL FIELD~~ GRAVITATION (2)

In NEWTONIAN GRAVITY

(1) $\vec{a} = -\vec{\nabla}\phi$: the acceleration of a particle is determined by the gradient of the gravitational field

(2) $\nabla^2\phi = 4\pi G\rho$: the gravitational field is solution of the Poisson equation where the mass density is the source term.

IN GENERAL RELATIVITY

(1) HOW DOES THE CURVATURE OF SPACETIME MANIFESTS ITSELF AS GRAVI

(2) HOW DOES MATTER CURVES SPACETIME DETERMINING THE

Let's just state the basic points of our journey so far: goals of 4-d.

→ we look at all the events in spacetime as a four-dimensional continuum which we parameterize with a set of four coordinates

$$\boxed{x^\mu \quad , \quad \mu = 0,1,2,3}$$

→ given any point P, the equivalence principle tells us that in the neighborhood of each P we can find a locally inertial frame such that

$$\boxed{g_{\mu\nu}\Big|_P = \eta_{\mu\nu}} \quad , \quad \boxed{\partial_\mu g_{\rho\sigma}\Big|_P = 0} \qquad (*)$$

→ there is a UNIQUE determinable metric structure over spacetime $g_{\mu\nu}$ such that (*) holds (ie which becomes locally minkowskian), so $g_{\mu\nu}$ must be pseudo-riemannian with signature $(-,+,+,+)$

~~→ free particles~~

→ GRAVITY IS UNIVERSAL: it affects all the particles in the same way; it is "SO UNIVERSAL" that it can be described more naturally as a feature of the background where matter move, and no longer as a propagating field ("INSTEAD BEING A PLAYER, GRAVITY HAS BECOME PART OF THE STAGE" at Wittoker)

(*) $\boxed{\text{free particles}}$ now are those subjected only to gravity

(*) free particles move along "straight lines" in curved space: GEODESICS:

$$x^\mu(s) \Rightarrow \frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0$$

WE MUST NOW COMBINE THIS KNOWLEDGE WITH THE LAWS OF PHYSICS THAT WE GATHER FROM SPECIAL RELATIVITY.

- sometimes, the effects of gravity are small, thus we can find a local description of spacetime where gravity does not appear, and we just take the laws of special relativity as they are

- the equivalence principle tells us how to modify the special relativity description to include gravity: JUST COVARIANTIZE THEM IN CURVED SPACE!

Thus the strategy is:

1. TAKE THE LAWS OF PHYSICS VALID IN INERTIAL FRAME IN MINKOWSKY SPACETIME

2. WRITE THEM IN A TENSORIAL FORM

3. WE ASSERT THAT THEIR RESULTS REMAIN VALID IN CURVED SPACE, THE CURVED SPACE VERSION WE GET:

MINIMAL COUPLING BETWEEN MATTER AND CURVATURE

$$\begin{cases} \to \eta_{\mu\nu} \longrightarrow g_{\mu\nu} \\ \to \partial_\mu \longrightarrow D_\mu \end{cases}$$

with this we are including all the information about the different geometry of spacetime

EXAMPLES:

1) ~~Let's consider~~ GEODESICS EQUATIONS

This procedure is exactly what we did when we derived the geodesics equation

→ tangent vector $u^\mu(s)$ constant along the path if

$$\frac{du^\mu}{ds} = \frac{d^2x^\mu}{ds^2} = 0$$

BUT $$\frac{d^2 x^\mu}{ds} = \frac{dx^\nu}{ds}\partial_\nu \frac{dx^\mu}{ds} = 0$$

$$\downarrow \text{COVARIANTIZE}$$

$$\frac{dx^\nu}{ds} D_\nu \frac{dx^\mu}{ds} = 0$$

$$\Downarrow$$

GEODESICS EQUATION $$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0$$

which is the relativistic generalization of the Newton eq.

2) ENERGY MOMENTUM CONSERVATION

$$\partial_\mu T^{\mu\nu} = 0 \Rightarrow D_\mu T^{\mu\nu} = 0$$

3) MAXWELL EQUATIONS.

$$\partial_\mu F^{\mu\nu} = J^\nu \Rightarrow D_\mu F^{\mu\nu} = J^\nu \Rightarrow \text{MINIMAL COUPLING}$$

BUT HOW DO WE KNOW THAT THESE EQUATIONS INDEED DESCRIBE GRAVITY?

$\Rightarrow$ WE SHOULD BE ABLE TO RECOVER THE RESULTS OF NEWTONIAN GRAVITY UNDER CERTAIN CONDITIONS

Let's consider the following Newtonian limit of minimally coupled gravity:

1) particles move slowly w.r.t. the speed of light.

2) the gravitational field is weak, can be considered as a perturbation of flat spacetime

3) fields are static: no time dependence.

Now let's implement these conditions in the geodesics of motion:

1) $\Rightarrow$ $\dfrac{dx^i}{d\tau} \ll \dfrac{dt}{d\tau}$

2) $\Rightarrow$ $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ $\qquad |h_{\mu\nu}| \ll 1$

3) $\Rightarrow$ $\partial_0 g_{\mu\nu} = 0$

With 1) the geodesics equations become

$$\frac{dx^\mu}{d\tau} D_\mu \frac{dx^\nu}{d\tau} \approx \frac{d^2 x^\nu}{d\tau^2} + \Gamma^\nu_{00} \left(\frac{dt}{d\tau}\right)^2 = 0$$

The relevant Christoffel symbol here is:

$$\Gamma^{\mu}_{00} = \frac{1}{2} g^{\mu\lambda} \left(\underbrace{\partial_0 g_{\lambda 0}}_{=0} + \underbrace{\partial_0 g_{0\lambda}}_{=0} - \partial_\lambda g_{00} \right) \quad \text{FROM (3)}$$

thus $\Gamma^{\mu}_{00} = -\frac{1}{2} g^{\mu\lambda} \partial_\lambda g_{00}$

But from (2) we get the inverse metric being

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$$

[NB: $g_{\mu\nu} g^{\nu\rho} = \delta_\mu^\rho$ and expand to 1st order in h]

where $h^{\mu\nu} = \eta^{\mu\rho} \eta^{\nu\sigma} h_{\rho\sigma}$

$$\Rightarrow \Gamma^{\nu}_{00} = -\frac{1}{2} \eta^{\nu\rho} \partial_\rho h_{00}$$

and the geodesics equation becomes

$$\frac{d^2 x^\mu}{d\tau^2} - \frac{1}{2} \eta^{\mu\lambda} \partial_\lambda h_{00} \left(\frac{dt}{d\tau} \right)^2 = 0$$

Let's separate

$\rightarrow$ TIME COMPONENT $\quad \frac{d^2 t}{d\tau} = \frac{1}{2} \eta^{0\lambda} \partial_\lambda h_{00} \left(\frac{dt}{d\tau} \right)^2$

$$\Rightarrow \frac{d^2 t}{d\tau} = -\frac{1}{2} \underbrace{\partial_0 h_{00}}_{=0} \left(\frac{dt}{d\tau} \right)^2$$

$$= 0$$

$\Rightarrow \frac{dt}{d\tau}$ is constant

→ $i \neq 0$ COMPONENTS: $$\frac{d^2 x^i}{d\tau^2} = \frac{1}{2}\eta^{i\lambda}\partial_\lambda h_{00}\left(\frac{dt}{d\tau}\right)^2$$

BUT $\eta^{i\lambda} = \delta^{ij}$

$$\Rightarrow \frac{d^2 x^i}{d\tau^2} = \frac{1}{2}\partial_i h_{00}\left(\frac{dt}{d\tau}\right)^2$$

IF WE CONVERT THE DERIVATIVES ON THE LHS INTO DERIVATIVES W.R.T ~~CORRPER~~ TIME

$$\frac{d}{d\tau} = \frac{dt}{d\tau}\frac{d}{dt}$$

$$\Rightarrow \boxed{\frac{d^2 x^i}{dt^2} = \frac{1}{2}\partial_i h_{00}}$$

This is exactly the Newton equation for gravitation upon identification

$$h_{00} = -2\,\phi(\vec{x}) \quad , \quad \phi = \text{is the Newt. potential}$$

$$\Rightarrow g_{00} = -\left(1 + 2\,\phi(\vec{x})\right) \quad (*)$$

THUS: IF THIS IDENTIFICATION HOLDS: THE WEAK FIELD LIMIT OF G.R IS NEWTONIAN GRAVITY!

⇒ HOW CAN WE PROVE THAT IT HOLDS?

WE NEED A SET OF EQUATIONS FOR THE GRAVITATIONAL FIELD WHERE (*) IS INDEED A SOLUTION

eg. for a body of mass M: $\phi = -\frac{GM}{r}$