THE EINSTEIN EQUATIONS

The Einstein equations give a comprehensive description on how the metric responds to matter and energy.

⇒ We can derive them by using some PLAUSIBILITY ARGUMENTS, and then verify that they reduce to known equations in some specific limit.

→ Let's start noticing that we want to find an equation which REPLACES the POISSON EQUATION FOR THE NEWTON POTENTIAL

$$\nabla^2\phi(\vec{x}) = 4\pi G\rho(\vec{x}) \longrightarrow ?$$

this is the laplacian in flat 3 dimensional space

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \underbrace{\vec{\nabla}\cdot\vec{\nabla}}_{\text{scalar product}}$$

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) =$$

If we write the three components as $\partial_i \equiv \frac{\partial}{\partial x^i}$

$$\nabla^2\phi \equiv \partial_i\partial_j\delta^{ij}\phi$$

remember that in the Newtonian approximation ϕ was ENTERING IN THE METRIC!

Thus we want some set of TENSORIAL (COVARIANT) EQUATION such that:

LHS = RHS

Tensor involving the second derivatives of the metric

Tensor describing the matter and energy content of spacetime, which generalize $\rho(\vec{x})$.

LET'S START FROM A SIMPLER CASE:

$\Rightarrow$ EMPTY SPACE $\Rightarrow$ NO MATTER AND ENERGY

$\Rightarrow$ RHS $= 0$

What do we put on the left hand side?

•) We know that the Riemann tensor provides a description of spacetime curvature $\Rightarrow$ must enter in some way!

•) Remember the equations for tidal forces in Newt. theory

$$\frac{d^2 \xi^i}{dt^2} = - \sum_j \frac{\partial^2 \phi(x)}{\partial x^i \partial x^j} \xi^j = - \sum_j K_{ij} \xi^j$$

$$\Rightarrow \nabla^2 \phi = \mathrm{Tr}\, K = \sum_i K_{ii} = 4\pi G \rho(\vec{x})$$

if $\rho = 0$ and no deviation $\Rightarrow \mathrm{Tr}\, K = 0$

But we know the relativistic extension of this equation!!!

IT'S THE EQUATION FOR GEODESIC DEVIATION!

NO DEVIATION

$$\frac{D^2 \xi^\mu}{Ds^2} = - R^\mu{}_{\nu\rho\alpha}\, \xi^\rho \frac{dx^\nu}{ds}\frac{dx^\alpha}{ds}$$

The analogy with what happens in the Newtonian case suggests that, if there is no deviation (ie in the vacuum):

$$R^\mu{}_{\nu\mu\alpha} \frac{dx^\nu}{ds}\frac{dx^\alpha}{ds} = 0 \qquad \text{VACUUM.}$$

(exactly as, in empty space in the Newtonian theory

$$\sum_i K_{ii} = 0 \quad)$$

But since this must hold for any representative path:

$$R^\mu{}_{\nu\mu\alpha} = 0$$

$$\Updownarrow$$

$$R_{\rho\nu\mu\alpha}\, g^{\rho\mu} = 0 \quad \Rightarrow \text{THIS IS THE RICCI TENSOR!}$$

THUS: EINSTEIN EQUATIONS IN THE VACUUM:

$$\boxed{R_{\nu\alpha} = 0} \quad \Rightarrow \text{FLAT (MINKOWSKY) SPACE IS A SOLUTION.}$$

set of $\frac{n(n+1)}{2}$ EQUATIONS., because $R_{\mu\nu} = R_{\nu\mu}$

if $n=4$: $\frac{n(n+1)}{2} = \frac{4(5)}{2} = 10$ equations for the 10 components of the metric!

It's easy to verify that it reduces to the Newtonian Poisson equation in the weak field limit!

Uniquely, in terms of the metric and in a locally inertial frame (remember that $\partial_\mu g_{\rho\nu}|_P = 0$)

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\Big(\partial_\beta\partial_\gamma g_{\alpha\delta} - \partial_\beta\partial_\delta g_{\alpha\gamma} + \\ - \partial_\alpha\partial_\gamma g_{\beta\delta} + \partial_\alpha\partial_\delta g_{\beta\gamma}\Big)$$

$$\Rightarrow R_{\beta\delta} = g^{\alpha\gamma} R_{\alpha\beta\gamma\delta} =$$

$$= \frac{1}{2}\Big(g^{\alpha\gamma}\partial_\beta\partial_\gamma g_{\alpha\delta} - g^{\alpha\gamma}\partial_\beta\partial_\delta g_{\alpha\gamma} + \\ - \partial_\alpha\partial_\gamma g_{\beta\delta} g^{\alpha\gamma} + g^{\alpha\gamma}\partial_\alpha\partial_\delta g_{\beta\gamma}\Big) =$$

$$= \frac{1}{2}\Big(\partial^\alpha\partial_\beta g_{\alpha\delta} - \cancel{\partial_\beta\partial_\delta \delta^\alpha_\alpha} + \\ - \Box g_{\beta\delta} + \partial^\gamma\partial_\delta g_{\beta\gamma}\Big) =$$

$$= \frac{1}{2}\Big(\partial^\alpha\partial_\beta g_{\alpha\delta} + \partial^\alpha\partial_\delta g_{\alpha\beta} - \Box g_{\beta\delta}\Big)$$

If you consider the weak, static fields approximation ⊛

⊛ from the geod. eq. $h_{00} = -2\phi \Rightarrow$ generalize:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \Rightarrow h_{\mu\nu} = -2\delta_{\mu\nu}\phi(\vec{x})$$

$$h_{\mu\nu} = \begin{pmatrix} -2\phi & & & \\ & -2\phi & \emptyset & \\ & \emptyset & -2\phi & \\ & & & -2\phi \end{pmatrix} \Rightarrow g_{\mu\nu} = \begin{pmatrix} -1-2\phi & & & \\ & 1-2\phi & \emptyset & \\ & \emptyset & 1-2\phi & \\ & & & 1-2\phi \end{pmatrix}$$

and the line element is

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(1+2\phi)dt^2 + (1-2\phi)(dx^2+dy^2+dz^2)$$

In this case

$$R_{tt} = -\nabla^2\phi$$

$$R_{\mu\nu} = -\delta_{\mu\nu}\nabla^2\phi$$

$$\Downarrow$$

$$\nabla^2\phi = 0$$

so $R_{tt} = 0 \Rightarrow \nabla^2\phi(\vec{x}) = 0$ ok!

If we are not in empty space?

We must find a way to generalize the notion of mass density!

This is given by the

STRESS-ENERGY TENSOR $T^{\mu\nu}$
(OR ENERGY-MOMENTUM TENSOR)

→ if you have a single relativistic particle, the four momentum p^μ contains all the information about the energy and the momentum of the particle

$$p^\mu = (E, \vec{p}) = (mc, m\vec{u})$$

→ But if we have an extended body, we cannot provide the 4-momentum of each particle which compose it. We need something which allows us to describe it as a continuum (for example, if we have a fluid, we must be able to describe macroscopically its - density

- pressure

- viscosity

- entropy.

This can be done providing:

→ macroscopic four-velocity U^μ

→ stress-energy tensor $T^{\mu\nu}$: symmetric (2,0)-tensor

↓

STRATEGY: WORK IN MINKOWSKY SPACE AND THEN MINIMALLY COUPLE TO GRAVITY

PHYSICAL DEFINITION : $T^{\mu\nu}$ is the flux of p^μ across a surface of constant x^ν (ie in the direction x^ν)

ep T^{00} : flux of p^0 (energy) in the $x^0 = t$ direction ⇒

⇒ IF WE ARE IN THE REST FRAME the only energy is the mass-energy density so

$$T_{00} = \rho(x)$$

⇒ in the same frame, we can separate "0" from "i" comp

$$\left(\begin{array}{c|c} T^{00} & T^{0i} \\ \hline T^{i0} & T^{ij} \end{array}\right)$$

T^{00} : mass-energy density

T^{0i} : flux of energy

T^{i0} : momentum density

T^{ij} : momentum flux : stress

T^{ij} is the stress exerted by the i-th component of the momentum in the j-th direction

Thus

→ diagonal terms T^{11}, T^{22}, T^{33} : x, y, z, components of the force exerted by a fluid element respectively in the x, y, z, direction (on a unit surface)

↓

PRESSURE p_i

→ off-diagonal terms : viscosity between the i, and j direction

We can consider a pair of examples quite common in cosmology

⇒ DUST (matter for cosmology st) : namely a collection of particles at rest with respect to each other in spacetime

⇒ they all move with the same velocity: $U^\mu(x)$

If we have a set of n of such particles per unit volume, all with mass m;

$$\rho(x) = mn$$

and in the rest frame all the particles are at rest with respect to the others, so there is no flux of energy, and no pressure, thus

$$T^{\mu\nu}_{dust} = \begin{pmatrix} \rho(x) & & & \\ & 0 & & \emptyset \\ & & 0 & \\ \emptyset & & & 0 \end{pmatrix}$$

The generalization to ANY FRAME can be done by noticing that in the rest frame in MINKOWSKY

$$U^\mu = (1,0,0,0)$$

$$p^\mu = (m, \vec{0}) \quad , \quad p^\mu = m U^\mu$$

and we can introduce $N^\mu = n U^\mu$ where U^μ is the 4 velocity

$$N^\mu_{\text{rest frame}} = (n, \vec{0})$$

thus we can write

$$T^{\mu\nu}_{\text{dust}} = p^\mu N^\mu = m n U^\mu U^\nu = \rho U^\mu U^\nu \circledast$$

which in the rest frame is exactly what we wrote above where in a generic spacetime $U^\mu = (U^0, 0, 0, 0)$ with $U^0 \neq 1$ but determined by $U^\mu U^\nu g_{\mu\nu} = -1$

$\Rightarrow$ PERFECT FLUID : is just a fluid which can be completely specified by:

•) ITS ENERGY DENSITY IN THE REST FRAME $\rho(x)$

•) THE PRESSURE IN THE REST FRAME (isotropic) p

(the fact that it's isotropic mean:

$$T^{ij} = 0 \quad \text{if } i \neq j$$

$$T^{ij} = p \quad \text{for any } i = j$$

In the rest frame

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \Rightarrow \text{MINKOWSKY} \quad T^{\mu\nu} = (\rho + p) U^\mu U^\nu + p \eta^{\mu\nu}$$

IN A GENERIC DIFFERENT FRAME $T^{\mu\nu} = (\rho + p) U^\mu U^\nu + p g^{\mu\nu}$

THE STRESS-ENERGY TENSOR IS CONSERVED!

⊗ In flat spacetime:

$$T^{\mu\nu}_{fluid} = (\rho + p) U^\mu U^\nu + p\eta^{\mu\nu} \Rightarrow \partial_\mu T^{\mu\nu}_{fluid} = 0$$

⊗ In curved spacetime

⊗ Repl. $\eta \to g$, then

$$\nabla_\mu T^{\mu\nu} = 0$$

This equation encloses the conservation of energy and of linear momentum!

Let's prove it in flat space!

Let's integrate $\partial_\mu T^{\mu\nu}$ over a generic volume V:

⊗ for $\nu = 0$: $\displaystyle\int_V \partial_\mu T^{\mu 0} d^3x = \int_V \partial_0 T^{00} d^3x + \int_V \partial_i T^{i0} d^3x$

$$= \underbrace{\frac{d}{dt}\left(\int_V T^{00} d^3x\right)}_{\text{TOTAL ENERGY}} + \underbrace{\int_S T^{0i} dS^i}_{\text{FLUX OF ENERGY THROUGH } S} = 0$$

$$\frac{dE}{dt} = -\int_S T^{0i} dS^i$$ CONSERVATION OF ENERGY

⊗ for $\nu = i$: $\displaystyle\int_V \partial_\mu T^{\mu i} = \int_V \partial_0 T^{0i} + \int_V \partial_j T^{ij} = 0$

$$\Rightarrow \frac{dp_i}{dt} = -\int_S T^{ij} dS^j$$

CONSERVATION OF LINEAR MOMENTUM

NOW: BACK TO THE EINSTEIN EQUATIONS:

Now we know that on the RHS we should put $T^{\mu\nu}$

→ WHAT DO WE PUT ON THE LHS?

→ Something with two indeces and symmetric

→ The Ricci Tensor? Let's try:

$$R^{\mu\nu} = K\, T^{\mu\nu}$$

where K is a constant.

⇒ Does it work? ⇒ NOT REALLY, BECAUSE if we derive it we get

$$D_\mu R^{\mu\nu} = K D_\mu T^{\mu\nu} = 0 \quad \text{, but, in general}$$

$$D_\mu R^{\mu\nu} \neq 0$$

⇒ We know that ∃! a tensor which is symmetric, contains the second derivatives of the metric, and it is conserved!

⇒ EINSTEIN TENSOR

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \,, \quad D_\mu G^{\mu\nu} = 0$$

EINSTEIN EQUATIONS CAN BE:

$$G_{\mu\nu} = K T_{\mu\nu} \Rightarrow R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = K T_{\mu\nu}$$

Another form can be obtained contracting both sides with the metric:

$$g^{\mu\nu}\left(R_{\mu\nu}-\frac{1}{2}R\,g_{\mu\nu}\right)=k\,g^{\mu\nu}T_{\mu\nu}$$

$$R-\frac{1}{2}R\cdot 4=k\,T$$

$$\Rightarrow \quad R=-kT$$

and replacing in the equation above:

$$R_{\mu\nu}+\frac{k}{2}T\,g_{\mu\nu}=k\,T_{\mu\nu}$$

$$\Rightarrow \quad R_{\mu\nu}=k\left(T_{\mu\nu}-\frac{1}{2}T\,g_{\mu\nu}\right)$$

$\Rightarrow$ <u>TO FIX THE CONSTANT</u> k let's consider the Newtonian approximation for a blob of dust.

$$T_{\mu\nu}=\rho(x)\,U_\mu U_\nu \xrightarrow{\text{REST FRAME}} U^\mu=(U^0,0,0,0)$$

thus

$$T^{\mu\nu}=\begin{pmatrix} (U^0)^2\rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

NEWT. LIMIT: $U^0\sim 1$

Let's remember that

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu} \qquad h_{\mu\nu}=-2\delta_{\mu\nu}\Phi$$

Let's compute T of the first order in the metric:

$$T = T_{\mu\nu}\, g^{\mu\nu} = T_{00}\, g^{00} = \rho(x)\left(-1 + h_{00}\right) \sim -\rho(x)$$

because $\rho(x)$ is already small in this weak field limit

Thus:
$$R_{00} = k\left(T_{00} - \frac{1}{2} T g_{00}\right) = k\left(\rho - \frac{1}{2}(-\rho)(-1)\right)$$
$$= k\left(\rho - \frac{1}{2}\rho\right) = \frac{k}{2}\rho$$

Let's evaluate R_{00}:

$R_{00} = R^{\mu}{}_{0\mu0}$ = only terms $\mu \equiv i$ contributes so

$$R^{i}{}_{0j0} = \partial_j \Gamma^{i}_{00} - \underbrace{\partial_0 \Gamma^{i}_{j0}}_{=0 \text{ for static fields}} + \underbrace{\cancel{\Gamma^{i}_{j\lambda}\Gamma^{\lambda}_{00}} - \cancel{\Gamma^{i}_{0\lambda}\Gamma^{\lambda}_{j0}}}_{\text{we can neglect them because they're higher order.}}$$

Thus

$$R^{i}{}_{0j0} \simeq \partial_j \Gamma^{i}_{00} \qquad \text{or} \qquad \Gamma^{i}_{00} = -\frac{1}{2}\partial^{i} g_{00}$$

and $R_{00} = \delta^{j}_{i} R^{i}{}_{0j0} \simeq \partial_i \Gamma^{i}_{00} =$ because static metric

$$= \partial_i\left[\frac{1}{2} g^{i\lambda}\left(\cancel{\partial_0 g_{0\lambda}} + \cancel{\partial_0 g_{\lambda 0}} - \partial_\lambda g_{00}\right)\right] =$$

$$= \frac{1}{2}\partial_i\left(-\cancel{g^{i0}\partial_0 g_{00}} - g^{ij}\partial_j g_{00}\right)$$

because $g^{i0} = 0$

$$= -\frac{1}{2}\partial_i\left[g^{ij}\partial_j g_{00}\right]$$

BUT $$g_{00} = \eta_{00} + h_{00} = -1 - 2\phi(x)$$

$$g^{ij} = \delta^{ij}$$

Then $$R_{00} \approx -\frac{1}{2}\partial_i \delta^{ij} \partial_j \left(-1 - 2\phi(x)\right)$$

$$R_{00} \approx \partial_i \partial^i \phi(x) = \nabla^2 \phi(x)$$

The einstein equation becomes

$$R_{00} \approx \nabla^2 \phi(x) = \frac{k}{2}\rho \Rightarrow$$

BUT THIS IS EXACTLY THE POISSON EQ. FOR THE Newt. GRAV. FIELD FOR

$$k = 8\pi G$$

$\Rightarrow$ THE EINSTEIN EQUATIONS ARE

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G\, T_{\mu\nu}$$

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

⇒ STRESS-ENERGY TENSOR: SOURCE FOR SPACETIME CURVATURE

⇒ NON LINEAR EQUATIONS: THE SUPERPOSITION PRINCIPLE DOES NOT HOLD: CANNOT WRITE A NEW SOL. AS LIN. COMB. OF 2 OTHERS.

⇒ In the weak field approximation the Einstein equations become

$$\Box h_{\mu\nu} = \text{const.}\, T_{\mu\nu}$$

empty space ⇒

$$\Box h_{\mu\nu} = 0$$

⇒ this is a wave equation! resembles $\Box A^\mu = 0$

⇓

EQUATION WHICH DESCRIBES THE PROPAGATION OF GRAVITATIONAL WAVES.

⇒ THE EINSTEIN EQUATIONS CAN BE COMPLETED WITH A COSMOLOGICAL CONSTANT TERM WHICH TAKE INTO ACCOUNT THE CONTRIBUTION OF A POSSIBLE VACUUM ENERGY DENSITY

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Λ is the cosmological constant

G.R: the source of the gravitational field is the full energy momentum tensor.

NEWT: only changes (differencies) between energy from different states are detectable

$$V(x) \text{ same physics as } V(x)+V_0$$

constant

In G.R. this V_0 would count!

⇒ open to the presence of a VACUUM ENERGY

⇒ ENERGY DENSITY OF EMPTY SPACE

should be homogeneous and isotropic and Lorentz invariant

$$T_{\mu\nu}^{(vac)} = -\underset{\text{constant}}{\rho_{vac}}\, g_{\mu\nu}$$

⇒ same as perfect fluid with isotropic pressure $P_{vac} = -\rho_{vac}$

EINST. EQ. BECOMES

$$\Rightarrow R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G \left(T_{\mu\nu}^{(M)} + T_{\mu\nu}^{vac} \right)$$

$$= 8\pi G \left(T_{\mu\nu}^{(M)} - \rho_{vac}\, g_{\mu\nu} \right)$$

BUT: EINSTEIN'S COSMOLOGICAL CONSTANT

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \Rightarrow \rho_{vac} = \frac{\Lambda}{8\pi G}$$

INTRODUCING COSMOLOGICAL CONSTANT ≡ VACUUM ENERGY DENSITY

BUT : there is still a mistery to be solved.

if we are in $d=4$, then ($\mu, \nu, \ldots = 0,1,2,3$),

$$\frac{n(n+1)}{2} = 10$$

Thus we have 10 equations, and the metric has 10 unknowns: BUT IS THIS TRUE?

Remember that the description of general relativity, the geometry that we are describing, is invariant under a change of coordinates: $x^\mu \to x'^\mu$, under which

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma g_{\rho\sigma}(x)$$

$\Downarrow$

NOT ALL THE COMPONENTS of g are unknown!

$\Downarrow$

WITHIN THE EINSTEIN EQUATION THERE MUST BE A SUBSET WHICH IS AUTHOMATICALLY VERIFIED:

$\Rightarrow$ THESE ARE THE BIANCHI IDENTITIES!

which are contained in the Einstein equation and comes from simply taking their covariant derivative

$$D_\mu \left(G^{\mu\nu} \right) = D_\mu \left(8\pi G T^{\mu\nu} \right) = 8\pi G D_\mu T^{\mu\nu} = 0$$

$$\boxed{D_\mu G^{\mu\nu} = 0}$$ BIANCHI IDENTITIES!