2C1: Further Maths, Handout 1

Niels R. Walet, September 29, 2002

1 ODEs: 1st order

- ODE $dy/dx = f(x)$ solved by direct integration $\dfrac{dy}{dx} = f(x) \rightsquigarrow y(x) = \displaystyle\int f(x)\,dx$
- $dy/dx = h(y)$ solved by inversion $\dfrac{dy}{dx} = h(y) \rightsquigarrow x(y) = \displaystyle\int 1/h(y)\,dy$
- $\frac{dy}{dx} = f(x)h(y)$ solved by separation of variables, $\dfrac{dy}{dx} = f(x)h(y) \rightsquigarrow \displaystyle\int \frac{1}{h(y)} dy = \int f(x)\,dx$
- Linear: $a(x)dy/dx + b(x)y = f(x)$ solved by finding integrating factor $p(x) = \int b(x)/a(x)\,dx$: $y(x) = e^{-p(x)} \displaystyle\int \frac{f(x)}{a(x)} e^{p(x)} dx$

2 Examples

Question: Find the solution of $y' = y^2 + 1$ satisfying the condition $y(0) = 1$.

Question: Find the general solution of $(y')^2 = 1 - y^2$.
Question: Solve $xy' + xy\ln(x) = e^{-(x-1)\ln(x)}$
Question: Solve $y' + \dfrac{y(x+y)}{x(x-y)} = 1$

3 ODEs: 2nd order

- Linear 2nd order equation with constant coefficients: $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = f(x)$. Homogeneous if $f(x) = 0$; inhomogenous otherwise
- simplest case: $\dfrac{d^2y}{dt^2} = f(t) \rightsquigarrow y(t) = \displaystyle\int \left(\int^t f(t')dt' \right) dt$.
- General homogeneous: substitute $y = e^{\lambda x}$: $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = 0 \rightsquigarrow a\lambda^2 + b\lambda + c = 0$.
- Driven (i.e., inhomogeneous): Solve homogeneous problem first, then add a particular solution to that.

4 Examples

Question: The DE

$$\frac{d^2x}{dt^2} = ge^{-kt}$$

(g acceleration due to gravity, k a positive constant) describes the position of a falling parachutist. "Classify" this equation and provide the general solution. Now assume that the initial position is h and velocity is zero.
Question: In a medium the z-component of the electric field varies with the distance x from the boundary as $\dfrac{d^2E_z}{dx^2} + \dfrac{\Omega^2}{c^2}\dfrac{dE_z}{dx} - \mu^2 E_z = 0$. Solve this equation, and determine the parameter ranges where this solution has a different character.
Question: Find the general solution to the DE ($\omega_0^2 \neq \Omega^2$) $\dfrac{d^2y}{dt^2} + \omega_0^2 y = A\cos(\Omega t)$.

5 What have we learned?

- What is a DE?
- Why do we concentrate on linear DE?
- What is the order of an ODE?
- What is the difference between homogenous and inhomogeneous equations in mathematical and physical terms?
- Why do integration constants appear? How many?