2C1: Further Maths, Handout 2

Niels R. Walet, September 29, 2002Niels.Walet@umist.ac.uk,
http://walet.phy.umist.ac.uk/2C1/

Exercises in class

Which of these equations is linear? and which is homogeneous?

a
$$\frac{\partial^2 u}{\partial x^2} + x^2 \frac{\partial u}{\partial y} = x^2 + y^2$$

b
$$y^2 \frac{\partial^2 u}{\partial x^2} + u \frac{\partial u}{\partial x} + x^2 \frac{\partial^2 u}{\partial y^2} = 0$$

c
$$\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial^2 u}{\partial y^2} = 0$$

What is the order of the following equations

a
$$\frac{\partial^3 u}{\partial x^3} + \frac{\partial^2 u}{\partial y^2} = 0$$

b
$$\frac{\partial^2 u}{\partial x^2} - 2 \frac{\partial^4 u}{\partial x^3 \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$$

Few examples:

- The wave equation, $\nabla^2 u = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$.
  This can be used to describes the motion of a string or drumhead (u is vertical displacement), as well as a variety of other waves (sound, light, ...). The quantity c is the speed of wave propagation.
- The heat or diffusion equation, $\nabla^2 u = \frac{1}{k} \frac{\partial u}{\partial t}$.
  This can be used to describe the change in temperature (u) in a system conducting heat, or the diffusion of one substance in another (u is concentration). The quantity k, sometimes replaced by a^2, is the diffusion constant, or the heat capacity. Notice the irreversible nature: If $t \to -t$ the wave equation turns into itself, but not the diffusion equation.
- Laplace's equation $\nabla^2 u = 0$.
- Helmholtz's equation $\nabla^2 u + \lambda u = 0$.
  This occurs for waves in wave quides, when searching for eigenmodes (resonances).
- Poisson's equation $\nabla^2 u = f(x, y, \ldots)$.
  The equation for the gravitational field inside a gravitational body, or the electric field inside a charged sphere.
- Time-independent Schrödinger equation, $\nabla^2 u = \frac{2m}{\hbar^2}[E - V(x, y, \ldots)]u = 0$.
  $|u|^2$ has a probability interpretation.
- Klein-Gordon equation $\nabla^2 u - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} + \lambda^2 u = 0$.
  Relativistic quantum particles,
  $|u|^2$ has a probability interpretation.

These are all second order differential equations. The order is defined as the highest derivative appearing in the equation.

Small Quiz
What is the order of the following equations

a
$$\frac{\partial^3 u}{\partial x^3} + \frac{\partial^2 u}{\partial y^2} = 0$$

b
$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^4 u}{\partial x^3 u} + \frac{\partial^2 u}{\partial y^2} = 0$$

Small Quiz
Classify the following differential equations (as elliptic, etc.)

a)
$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$$

b)
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} = 0$$

c)
$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x} = 0$$

d)
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$$

a) $a = b = c = 1$, $\Delta = 0$, parabolic
b) $a = b = 1$, $c = 0$, $\Delta = 1$, elliptic
c) $a = -b = 1$, $c = 0$, $\Delta = -1$, hyperbolic
d) $a = 1$, $b = c = 0$, $\Delta = 0$, parabolic