2C1=P303: P.D.E., Handout 4

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1 Fourier Series

A Fourier series is a series of the form Consider the series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)] \quad -L \le x \le L.$$

1.1 Periodic functions

A function is called periodic with period p if $f(x+p) = f(x)$, for all x. The smallest positive value of p for which f is periodic is called the (primitive) period of f.

1.2 Orthogonality and normalisation

The basic functions in a Fourier series are

$$\{1, \cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\},$$

We find that

$$\begin{aligned}
\int_{-L}^{L} 1\cdot 1 dx &= 2L,\\
\int_{-L}^{L} 1\cdot \cos\left(\frac{n\pi x}{L}\right) dx &= 0,\\
\int_{-L}^{L} 1\cdot \sin\left(\frac{n\pi x}{L}\right) dx &= 0,\\
\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\cdot \cos\left(\frac{n\pi x}{L}\right) dx &= \tfrac{1}{2}\int_{-L}^{L} \cos\left(\frac{(m+n)\pi x}{L}\right) + \cos\left(\frac{(m-n)\pi x}{L}\right) dx\\
&= \begin{cases} 0 & n \neq m \\ L & n = m \end{cases}\\
\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\cdot \sin\left(\frac{n\pi x}{L}\right) dx &= \tfrac{1}{2}\int_{-L}^{L} -\cos\left(\frac{(m+n)\pi x}{L}\right) + \cos\left(\frac{(m-n)\pi x}{L}\right) dx\\
&= \begin{cases} 0 & n \neq m \\ L & n = m \end{cases}\\
\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\cdot \sin\left(\frac{n\pi x}{L}\right) dx &= \tfrac{1}{2}\int_{-L}^{L} \sin\left(\frac{(m+n)\pi x}{L}\right) + \sin\left(\frac{(m-n)\pi x}{L}\right) dx\\
&= 0.
\end{aligned}$$

These functions orthogonal! The general definition of inner product is

$$(f,g) = \int_a^b w(x)f(x)g(x)dx.$$

If this is zero we say that the functions f and g are orthogonal on the interval $[a,b]$ with weight functions w.

The norm is defined as the square root of the inner-product of a function with itself

$$||f|| = \sqrt{\int_a^b w(x)f(x)^2 dx}.$$

If we define a normalised form of f (like a unit vector) as $f/||f||$, we have

$$||(f/||f||)|| = \sqrt{\frac{\int_a^b w(x)f(x)^2dx}{||f||^2}} = \frac{\sqrt{\int_a^b w(x)f(x)^2dx}}{||f||} = 1.$$

1.3 When is it a Fourier series?

Assume that the periodic function f has a Fourier series representation, Use the orthogonality of the trigonometric functions to find that

$$\begin{aligned}\int_{-L}^{L} f(x)\cdot 1 dx &= La_0,\\ \int_{-L}^{L} f(x)\cdot\cos\left(\frac{n\pi x}{L}\right)dx &= La_n,\\ \int_{-L}^{L} f(x)\cdot\sin\left(\frac{n\pi x}{L}\right)dx &= Lb_n.\end{aligned}$$

This defines the Fourier coefficients for a given f.

An important property of Fourier series is given in Parseval's lemma:

$$\int_{-L}^{L} f(x)^2 dx = \frac{La_0^2}{2} + L\sum_{n=1}^{\infty}(a_n^2+b_n^2).$$

Square wave

$$f(x) = \begin{cases} -3 & \text{if } -5+10n < x < 10n \\ 3 & \text{if } 10n < x < 5+10n \end{cases},$$

where n is an integer. $L = 5$.

$$\begin{aligned} a_0 &= \frac{1}{5}\int_{-5}^{0} -3dx + \frac{1}{5}\int_{0}^{5} 3dx = 0\\ a_n &= \frac{1}{5}\int_{-5}^{0} -3\cos\left(\frac{n\pi x}{5}\right) + \frac{1}{5}\int_{0}^{5} 3\cos\left(\frac{n\pi x}{5}\right) = 0\\ b_n &= \frac{1}{5}\int_{-5}^{0} -3\sin\left(\frac{n\pi x}{5}\right) + \frac{1}{5}\int_{0}^{5} 3\sin\left(\frac{n\pi x}{5}\right)\\ &= \frac{3}{n\pi}\cos\left(\frac{n\pi x}{5}\right)\Big|_{-5}^{0} - \frac{3}{n\pi}\cos\left(\frac{n\pi x}{5}\right)\Big|_{0}^{5}\\ &= \frac{6}{n\pi}[1-\cos(n\pi)] = \frac{12}{n\pi}\delta_{n,\text{odd}} \end{aligned}$$

And thus ($n = 2m+1$)

$$f(x) = \frac{12}{\pi}\sum_{m=0}\frac{1}{2m+1}\sin\left(\frac{(2m+1)\pi x}{5}\right).$$

Question: What happens if we apply Parseval's theorem to this series?
Answer: We find

$$\int_{-5}^{5} 9dx = 5\frac{144}{\pi^2}\sum_{m=0}^{\infty}\left(\frac{1}{2m+1}\right)^2$$

Which can be used to show that

$$\sum_{m=0}^{\infty}\left(\frac{1}{2m+1}\right)^2 = \frac{\pi^2}{8}.$$

1.4 Fourier series for even and odd functions

A function is called even if $f(-x) = f(x)$, e.g. $\cos(x)$.
A function is called odd if $f(-x) = -f(x)$, e.g. $\sin(x)$.

1. The sum of two even (odd) functions is even (odd).
2. The product of two even or two odd functions is even.
3. The product of an even and an odd function is odd.

Question: Which of the following functions is even or odd?
a) $\sin(2x)$, b) $\sin(x)\cos(x)$, c) $\tan(x)$, d) x^2, e) x^3, f) $|x|$

Answer: even: d, f; odd a, b, c, e.

Now if we look at the infinite Fourier series, the *Fourier cosine series*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x$$

describes an even function (why?), and the *Fourier sine series*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x$$

an odd function. These play an especially important rôle for functions defined on half the Fourier interval, i.e., on $[0, L]$ instead of $[-L, L]$. There are three possible ways to define a Fourier series in this way:

1. Continue f as an even function.

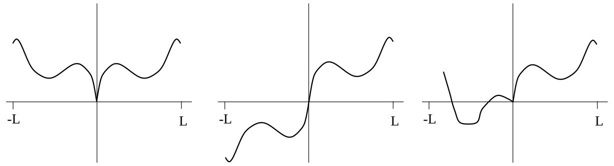

Figure 1: A sketch of the possible ways to continue f beyond its definition region for $0 < x < L$. From left to right as even function, odd function or assuming no symmetry at all.

2. Continue f as an odd function.
3. Neither of the two above.

A Fourier cosine series has $df/dx = 0$ at $x = 0$, and the Fourier sine series has $f(x = 0) = 0$. Let me check the first of these statements:

$$\frac{d}{dx}\left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x\right] = -\frac{\pi}{L}\sum_{n=1}^{\infty} n a_n \sin\frac{n\pi}{L}x = 0 \text{ at } x = 0.$$

As an example look at the function $f(x) = 1-x$, $0 \le x \le 1$, with an even continuation on the interval $[-1, 1]$. We find

$$\begin{aligned} a_0 &= \frac{2}{1}\int_0^1 (1-x)dx = 1 \\ a_n &= 2\int_0^1 (1-x)\cos n\pi x dx \\ &= \left\{\frac{2}{n\pi}\sin n\pi x - \frac{2}{n^2\pi^2}[\cos n\pi x + n\pi x \sin n\pi x]\right\}\Bigg|_0^1 \\ &= \begin{cases} 0 \text{ if } n \text{ even} \\ \frac{4}{n^2\pi^2} \text{ if } n \text{ odd} \end{cases} \end{aligned}$$

So

$$f(x) = \tfrac{1}{2} + \frac{4}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\cos(2m+1)\pi x.$$

1.5 Convergence of Fourier series

1. A square wave, $f(x) = 1$ for $-\pi < x < 0$; $f(x) = -1$ for $0 < x < \pi$.

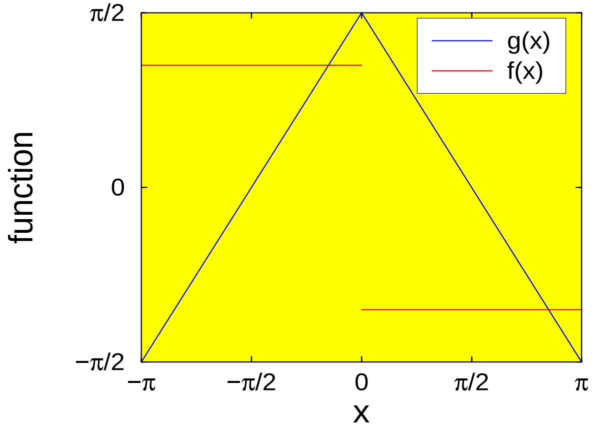

Figure 2: The square and triangular waves on their fundamental domain.

2. a triangular wave, $g(x) = \pi/2 + x$ for $-\pi < x < 0$; $g(x) = \pi/2 - x$ for $0 < x < \pi$.

Note that f is the derivative of g.

It is not very hard to generate the relevant Fourier series,

$$\begin{aligned} f(x) &= -\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{2m+1}\sin(2m+1)x, \\ g(x) &= \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\cos(2m+1)x. \end{aligned}$$

Let us compare the partial sums, where we let the sum in the Fourier series run from zero to M i.o. ∞.

The convergence for g is uneventful, and after a few steps it is hard to see a difference between the partial sums, as well as between the partial sums and g. For f, the square wave, we see a surprising result: Even though the approximation gets better and better in the flat middle, there is a finite (and constant!) overshoot near the singularity. The area of this overshoot becomes smaller and smaller as we increase M. This is called the Gibbs phenomenon (after its discoverer). It can be shown that for any function

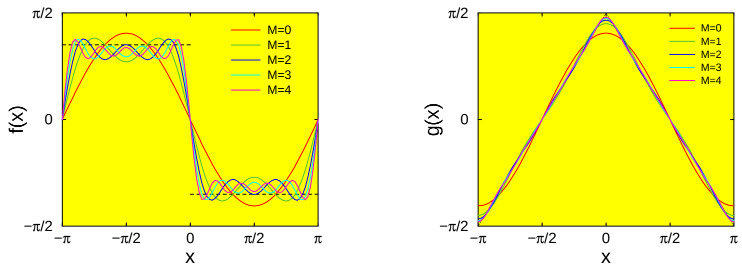

Figure 3: The convergence of the Fourier series for the square (left) and triangular wave (right). the number M is the order of the highest Fourier component.

with a discontinuity such an effect is present, and that the size of the overshoot only depends on the size of the discontinuity! Let me show a final, slightly more interesting version of this picture.

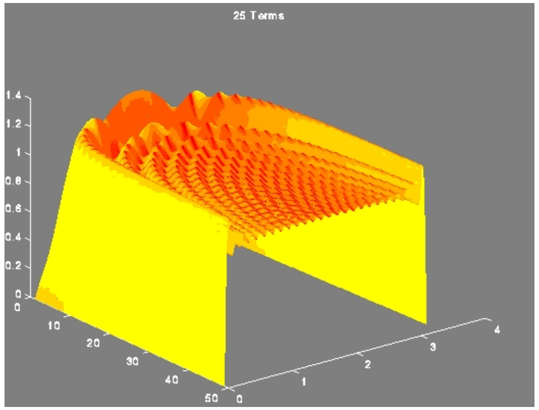

Figure 4: A three-dimensional representation of the Gibbs phenomemnon for the square wave. The axis orthogonal to the paper labels the number of Fourier components.