2C1: P.D.E., Handout 5

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1 Cookbook for separation of variables

- Take care that the boundaries are naturally described in your variables (i.e., at the boundary one of the coordinates is constant)!
- Write the unknown function as a product of functions in each variable.
- Divide the differential equation by the product of functions, so as to have a ratio of functions in one variable equal to a ratio of functions in the other variable(s).
- Since these two are equal they must both equal to a constant.
- Separate the boundary and initial conditions. Those that are zero can be re-expressed as conditions on one of the unknown functions.
- Solve the equation for that function where most boundary information is known.
- This determines a (discrete) set of separation parameters.
- Solve the remaining equation for each allowed value of the separation parameter.
- Use the superposition principle (which holds for linear homogeneous equations) to add all solutions, each multiplied by a constant.
- Determine the value of these constants from the remaining boundary and initial conditions.

This is best illustrated with an example:

2 parabolic equation*

Heat equation in 1 space dimension.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},\ \ 0 < x < L,\ t > 0.$$

BC's

$$u(0,t) = 0,\ u(L,t) = 0,\ \ t > 0$$

IC

$$u(x,0) = x,\ 0 < x < L.$$

Use separation of variables:

$$u(x,t) = X(x)T(t).$$

This leads to the differential equation

$$X(x)T'(t) = kX''(x)T(t).$$

We find, by dividing both sides by XT, that

$$\frac{1}{k}\frac{T'(t)}{T(t)} = \frac{X"(k)}{X(k)}.$$

Thus the left-hand side, a function of t, equals a function of x on the right-hand side. This is not possible unless both sides are independent of x and t, i.e. constant. Let us call this constant $-\lambda$.

Question: What happens if $X(x)T(t)$ is zero at some point?

Answer: Nothing. We can still obtain the same answer, even though we can't divide.

The boundary conditions also separate (only for zero r.h.s.!!!)

$$\begin{aligned} u(0,t) = 0 \quad &\rightarrow \quad X(0)T(t) = 0 \rightarrow X(0) = 0 \\ u(L,t) = 0 \quad &\rightarrow \quad X(L)T(t) = 0 \rightarrow X(0) = 0 \end{aligned}$$

The initial condition will not separate,

$$u(x,0) = X(x)T(0) = x,$$

can not be solve for $X(x)$ or $T(0)$ since the r.h.s. is not zero.

We obtain two differential equations

$$T'(t) \quad = \quad -\lambda kT(t), X"(x) = -\lambda X(x).$$

We now have to distinguish the three cases $\lambda > 0$, $\lambda = 0$, $\lambda < 0$.

$\boxed{\lambda > 0}$

Write $\alpha^2 = \lambda$. The solution to the equation for X is

$$X(x) = A\cos\alpha x + B\sin\alpha x.$$

$X(0) = 0$ gives $A \cdot 1 + B \cdot 0 = 0$, or $A = 0$. Using $X(L) = 0$ we find that

$$B\sin\alpha L = 0$$

which has a "nontrivial" solution only when $\alpha L = n\pi$. This gives $\lambda_n = \frac{n^2\pi^2}{L^2}$.

$\boxed{\lambda = 0}$

We find that $X = A + Bx$. The boundary conditions give $A = B = 0$, so only trivial (zero) solution.

$\boxed{\lambda < 0}$

We write $\lambda = -\alpha^2$. The solution for X is now in term of exponential, or hyperbolic functions,

$$X(x) = A \cosh x + B \sinh x.$$

The boundary condition at $x = 0$ gives $A = 0$, and the one at $x = L$ gives $B = 0$. Only trivial solution.

We have thus only found a solution for $\lambda > 0$. Solving the T equation, we find $T = \exp(-\lambda kT)$. Combining the solutions, we have

$$u_n(x,t) = \exp\left(-k\frac{n^2\pi^2}{L^2}t\right)\sin\frac{n\pi}{L}x.$$

The equation we started from was linear and homogeneous, so we can superimpose the solutions for different values of n,

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} c_n \exp\left(-k\frac{n^2\pi^2}{L^2}t\right)\sin\frac{n\pi}{L}x.}$$

This can be thought of as a Fourier sine series with time-dependent Fourier coefficients. The initial condition specifies the coefficients c_n, which are the Fourier coefficients at time $t = 0$. Thus

$$\begin{aligned} c_n &= \frac{2}{L}\int_0^L x \sin\frac{n\pi x}{L}dx \\ &= -\frac{2L}{n\pi}(-1)^n = (-1)^{n+1}\frac{2L}{n\pi} \end{aligned}$$

The final solution to the PDE + BC's + IC is

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2L}{n\pi}\exp\left(-k\frac{n^2\pi^2}{L^2}t\right)\sin\frac{n\pi}{L}x.}$$

This solution is transient: if time goes to infinity, it goes to zero.

3 hyperbolic equation*

As an example of a hyperbolic equation study the wave equation. One of the systems it can describe is a transmission line for high frequency signals, 40m long.

$$\begin{aligned} \frac{\partial^2 V}{\partial x^2} &= \underbrace{LC}_{imp\times capac}\frac{\partial^2 V}{\partial t^2} \\ \frac{\partial V}{\partial x}(0,t) &= \frac{\partial V}{\partial x}(40,t) = 0 \\ V(x,0) &= f(x) \\ \frac{\partial V}{\partial t}(x,0) &= 0 \end{aligned}$$

4 Laplace's equation*

Solve Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

an example of an *elliptic* equation. Look at a square plate of size $a \times b$, and impose the boundary conditions

$$\begin{aligned} u(x,0) &= 0, \\ u(a,y) &= 0, \\ u(x,b) &= x, \\ u(0,y) &= 0 \end{aligned}$$

(This choice is made so as to be able to evaluate Fourier series easily. It is not very realistic!)

5 More complex initial/boundary conditions

It is not always possible on separation of variables to separate initial or boundary conditions in a condition on one of the two functions. We can either map the problem into simpler ones by replacing using superposition of boundary conditions, or we can carry around additional integration constants.

Let me give an example of this procedure. Consider a vibrating string attached on two air bearings, along rods 4m apart. Now we are asked to find the displacement

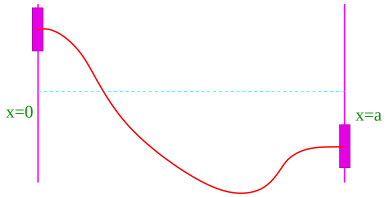

Figure 1: A string connected to air bearings.

if the initial displacement is one meter and the initial velocity is x/t_0 m/s. The differential equation and its boundary conditions are easily written down,

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \\
\frac{\partial u}{\partial x}(0,t) &= \frac{\partial u}{\partial x}(4,t)+0, \quad \text{if } t>0 \\
u(x,0) &= 1 \\
\frac{\partial u}{\partial t}(x,0) &= x/t_0
\end{aligned}$$

Question: What happens if I add two solutions v and w of the differential equation that satisfy the same BC's as above but different IC's,

$$\begin{aligned}
v(x,0)=0, &\qquad \frac{\partial v}{\partial t}(x,0)=x/t_0, \\
w(x,0)=1, &\qquad \frac{\partial w}{\partial t}(x,0)=0?
\end{aligned}$$

Answer: u=$v+w$, we can add the BC's.

If we separate variables, $u(x,t)=X(x)T(t)$, we find that we obtain simple boundary conditions for $X(x)$,

$$X'(0)=X'(4)=0,$$

but we have no such luck for $T(t)$. As before we solve the eigenvalue equation for X, and find solutions $\lambda_n=\frac{n^2\pi^2}{16}$, $n=0,1,...$, and $X_n(x)=\cos(\frac{n\pi}{4}x)$. Since we have no boundary conditions for $T(t)$, we have to take the full solution,

$$\begin{aligned}
T_0(t) &= A_0+B_0t, \\
T_n(t) &= A_n\cos\frac{n\pi}{4}ct+B_n\sin\frac{n\pi}{4}ct,
\end{aligned}$$

and thus

$$u(x,t)=\tfrac{1}{2}(A_0+B_0t)+\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi}{4}ct+B_n\sin\frac{n\pi}{4}ct\right)\cos\frac{n\pi}{4}x.$$

Now impose the initial conditions:

$$u(x,0)=1=\tfrac{1}{2}A_0+\sum_{n=1}^{\infty}A_n\cos\frac{n\pi}{4}x.$$

Which implies $A_0=2$, $A_n=0, n>0$. And

$$\frac{\partial u}{\partial t}(x,0)=x/t_0=\tfrac{1}{2}B_0+\sum_{n=1}^{\infty}\frac{n\pi c}{4}.B_n\cos\frac{n\pi}{4}x$$

This is the sine Fourier series of x, which we have encountered before, and leads to the coefficients $B_0=4$ and $B_n=-\frac{64}{n^3\pi^3c}$ if n is odd and zero otherwise. So finally

$$u(x,t)=(1+2t)-\frac{64}{\pi^3}\sum_{n\ \text{odd}}\frac{1}{n^3}\sin\left(\frac{n\pi ct}{4}\right)\cos\left(\frac{n\pi x}{4}\right).$$

We could also have treated the problem differently, by finding two solutions to the wave equation, one with the initial conditions $u(x,0)=1$, $\frac{\partial u}{\partial t}(x,0)=0$ and the other with $u(x,0)=0$, $\frac{\partial u}{\partial t}(x,0)=x$. This is a very general technique (and maybe overkill for the present problem), but allows for solutions with only one set of integration constants.

6 Inhomogeneous equations

Consider a rod of length 2m, laterally insulated (heat only flows inside the rod). Initially the temperature u is

$$\frac{4}{\pi^2k}\sin\left(\frac{\pi x}{2}\right)+500\text{ K}.$$

(The weird choice are so as to have an easy solution!) The left and right ends are both attached to a thermostat, and the temperature at the left side is fixed at a temperature of 500 K and the right end at 100 K. There is also a heater attached to the rod that adds a constant heat of $\sin\left(\frac{\pi x}{2}\right)$ to the rod. The differential equation describing this is inhomogeneous:

$$\begin{aligned}\frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} + \sin\left(\frac{\pi x}{2}\right),\\ u(0,t) &= 500,\\ u(\pi,t) &= 100,\\ u(x,0) &= \frac{1}{k}\sin\left(\frac{\pi x}{2}\right) + 500.\end{aligned}$$

Since the inhomogeneity is time-independent we write

$$u(x,t) = v(x,t) + h(x),$$

where h will be determined so as to make v satisfy a homogeneous equation. Substituting this form, we find

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + kh'' + \sin\left(\frac{\pi x}{2}\right).$$

To make the equation for v homogeneous we require

$$h''(x) = -\frac{1}{k}\sin\left(\frac{\pi x}{2}\right),$$

which has the solution

$$h(x) = C_1 x + C_2 + \frac{4}{k\pi^2}\sin\left(\frac{\pi x}{2}\right).$$

At the same time we let h carry the boundary conditions, $h(0) = 500$, $h(2) = 100$, and thus

$$h(x) = -200x + 500 + \frac{4}{k\pi^2}\sin\left(\frac{\pi x}{2}\right).$$

The function v satisfies

$$\begin{aligned}\frac{\partial v}{\partial t} &= k\frac{\partial^2 v}{\partial x^2},\\ v(0,t) &= v(\pi,t) = 0,\\ v(x,0) &= u(x,0) - h(x) = 200x.\end{aligned}$$

This is a problem of a type that we have seen before. By separation of variables we find

$$v(x,t) = \sum_{n=1}^{\infty} b_n \exp(-k(n\pi/2)^2 t)\sin n\pi x.$$

The initial condition gives

$$\sum_{n=1}^{\infty} b_n \sin nx = \frac{400}{\pi}x.$$

from which we find

$$b_n = (-1)^{n+1}\frac{800}{n\pi}.$$

And thus

$$u(x,t) = -200x + 500\frac{4}{k\pi^2}\sin\left(\frac{\pi x}{2}\right). + \frac{800}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{\pi n x}{2}\right)e^{-kn^2 t}. \qquad (1)$$

Note: as $t \to \infty$, $u(x,t) \to -\frac{400}{\pi}x + 500 + \frac{\sin x}{k}$. As can be seen in Fig. 2 this approach is quite rapid – we have chosen $k = 1/100$ in that figure, and summed over the first 100 solutions.

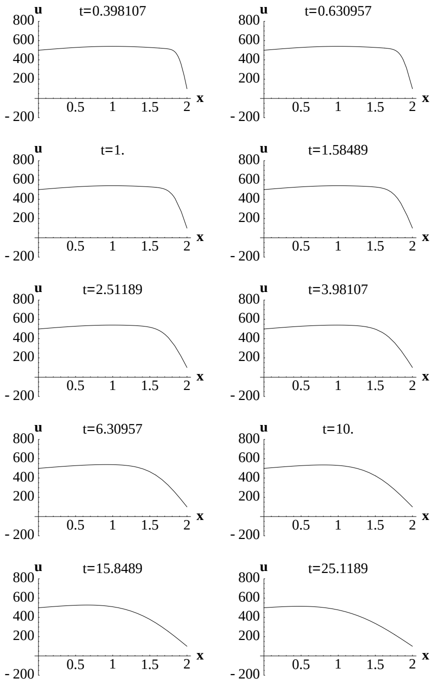

Figure 2: Time dependence of the solution to the inhomogeneous equation (1)