2C1: P.D.E., Handout 6

Niels R. Walet, November 11, 2002
Niels.Walet@umist.ac.uk, **http://walet.phy.umist.ac.uk/2C1/**

1 Polar coordinates

Polar coordinates in two dimensions are defined by

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi,$$
$$\rho = \sqrt{x^2+y^2}, \qquad \phi = \arctan(y/x),$$

as indicated schematically in Fig. 1.

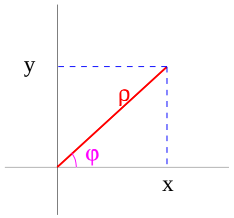

Figure 1: Polar coordinates

Using the chain rule we find

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial\rho}{\partial x}\frac{\partial}{\partial\rho} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi} \\
&= \frac{x}{\rho}\frac{\partial}{\partial\rho} - \frac{y}{\rho^2}\frac{\partial}{\partial\phi} = \cos\phi\frac{\partial}{\partial\rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi}, \\
\frac{\partial}{\partial y} &= \frac{\partial\rho}{\partial y}\frac{\partial}{\partial\rho} + \frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi} \\
&= \frac{y}{\rho}\frac{\partial}{\partial\rho} + \frac{x}{\rho^2}\frac{\partial}{\partial\phi} = \sin\phi\frac{\partial}{\partial\rho} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial\phi},
\end{aligned}$$

We can write

$$\nabla = \hat{\mathbf{e}}_\rho\frac{\partial}{\partial\rho} + \hat{\mathbf{e}}_\phi\frac{1}{\rho}\frac{\partial}{\partial\phi}$$

where the unit vectors

$$\begin{aligned}
\hat{\mathbf{e}}_\rho &= (\cos\phi, \sin\phi), \\
\hat{\mathbf{e}}_\phi &= (-\sin\phi, \cos\phi),
\end{aligned}$$

are an orthonormal set. We say that circular coordinates are *orthogonal*.

We can now use this to evaluate ∇^2,

$$\begin{aligned}
\nabla^2 &= \cos^2\phi\frac{\partial^2}{\partial\rho^2} + \frac{\sin\phi\cos\phi}{\rho^2}\frac{\partial}{\partial\phi} + \frac{\sin^2\phi}{\rho}\frac{\partial}{\partial\rho} + \frac{\sin^2\phi}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\sin\phi\cos\phi}{\rho^{`2}}\frac{\partial}{\partial\phi} \\
&\quad + \sin^2\phi\frac{\partial^2}{\partial\rho^2} - \frac{\sin\phi\cos\phi}{\rho^2}\frac{\partial}{\partial\phi} + \frac{\cos^2\phi}{\rho}\frac{\partial}{\partial\rho} + \frac{\cos^2\phi}{\rho^2}\frac{\partial^2}{\partial\phi^2} - \frac{\sin\phi\cos\phi}{\rho^2}\frac{\partial}{\partial\phi} \\
&= \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} \\
&= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}.
\end{aligned}$$

A final useful relation is the integration over these coordinates.

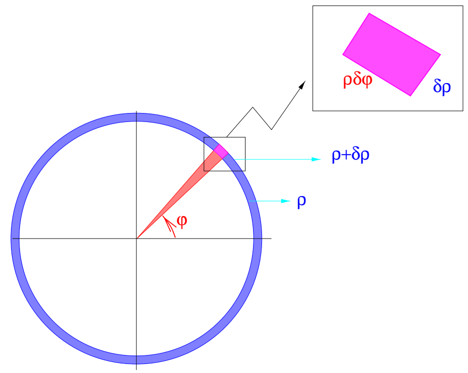

Figure 2: Integration in polar coordinates

As indicated schematically in Fig. 2, the surface related to a change $\rho \to \rho + \delta\rho$, $\phi \to \phi + \delta\phi$ is $\rho\delta\rho\delta\phi$. This leads us to the conclusion that an integral over x, y can be rewritten as

$$\int_V f(x,y)dxdy = \int_V f(\rho\cos\phi, \rho\sin\phi)\rho d\rho d\phi \tag{1}$$

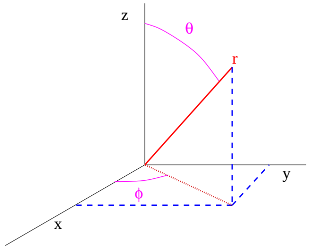

Figure 3: Spherical coordinates

2 spherical coordinates

Spherical coordinates are defined as

$$x = r\cos\phi\sin\theta, \qquad y = r\sin\phi\sin\theta,\ z = r\cos\theta,$$
$$r = \sqrt{x^2+y^2+z^2}, \qquad \phi = \arctan(y/x),\ \theta = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right),$$

as indicated schematically in Fig. 3.

Using the chain rule we find

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} \\
&= \frac{x}{r}\frac{\partial}{\partial r} - \frac{y}{x^2+y^2}\frac{\partial}{\partial \phi} + \frac{xz}{r^2\sqrt{x^2+y^2}}\frac{\partial}{\partial \theta} \\
&= \sin\theta\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial \phi} + \frac{\cos\phi\cos\theta}{r}\frac{\partial}{\partial \theta}, \\
\frac{\partial}{\partial y} &= \frac{y}{r}\frac{\partial}{\partial r} + \frac{x}{x^2+y^2}\frac{\partial}{\partial \phi} + \frac{yz}{r^2\sqrt{x^2+y^2}}\frac{\partial}{\partial \theta} \\
&= \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial \phi} + \frac{\sin\phi\cos\theta}{r}\frac{\partial}{\partial \theta}, \\
\frac{\partial}{\partial z} &= \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial \phi} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} \\
&= \sin\theta\sin\phi\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}.
\end{aligned}$$

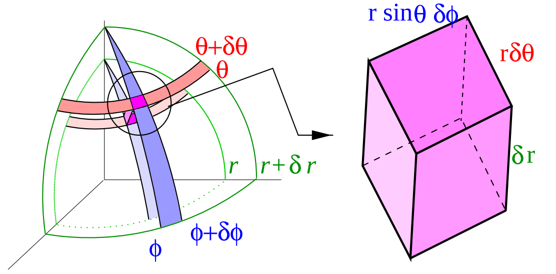

Figure 4: Integration in spherical coordinates

Once again we can write ∇ in terms of these coordinates,

$$\nabla = \hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \hat{\mathbf{e}}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} + \hat{\mathbf{e}}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}$$

$$\begin{aligned}
\hat{\mathbf{e}}_r &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \\
\hat{\mathbf{e}}_\phi &= (-\sin\phi, \cos\phi, 0), \\
\hat{\mathbf{e}}_\theta &= (\cos\phi\cos\theta, \sin\phi\cos\theta, -\sin\theta).
\end{aligned}$$

are an orthonormal set. We say that spherical coordinates are *orthogonal*.

We can use this to evaluate $\Delta = \nabla^2$,

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{2}$$

Finally, for integration over these variables we need to know the volume of the small cuboid contained between r and $r+\delta r$, θ and $\theta+\delta\theta$ and ϕ and $\phi+\delta\phi$. The length of the sides due to each of these changes is δr, $r\delta\theta$ and $r\sin\theta\delta\theta$, respectively. We thus conclude that

$$\int_V f(x,y,z)dxdydz = \int_V f(r,\theta,\phi)r^2\sin\theta drd\theta d\phi. \tag{3}$$