2C1: P.D.E., Handout 7

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Series solutions of D.E. (Frobenius' method)

1 A simple example

Look at differential equation,

$$y''(t) = t\, y(t),$$

with initial conditions $y(0) = a$, $y'(0) = b$. Let us assume that there is a solution that is analytical near $t = 0$. This means that near $t = 0$ the function has a Taylor's series

$$y(t) = c_0 + c_1 t + \ldots = \sum_{k=0}^{\infty} c_k t^k.$$

We find

$$\begin{aligned}
y'(t) &= c_1 + 2c_2 t + \ldots = \sum_{k=1}^{\infty} k c_k t^{k-1} \\
y''(t) &= 2c_2 + 3\cdot 2t + \ldots = \sum_{k=2}^{\infty} k(k-1) c_k t^{k-1} \\
ty(t) &= c_0 t + c_1 t^2 + \ldots = \sum_{k=0}^{\infty} c_k t^{k+1} \\
y'' - ty &= [2c_2 + 3\cdot 2t + \ldots] - [c_0 t + c_1 t^2 + \ldots] \\
&= 2c_2 + (3\cdot 2c_3 - c_0)t + \ldots \\
&= 2c_2 + \sum_{k=3}^{\infty} \{k(k-1)c_k - c_{k-3}\}\, t^{k-2}.
\end{aligned}$$

Here we have collected terms of equal power of t. The reason is simple. We are requiring a power series to equal 0. The only way that can work is if each power of x in the powerseries has zero coefficient. (Compare a finite polynomial....) We thus find

$$c_2 = 0, \quad k(k-1)c_k = c_{k-3}.$$

The last relation is called a recurrence of recursion relation, which we can use to bootstrap from a given value, in this case c_0 and c_1. Once we know these two numbers, we can determine c_3,c_4 and c_5:

$$c_3 = \frac{1}{6}c_0, \quad c_4 = \frac{1}{12}c_1, \quad c_5 = \frac{1}{20}c_2 = 0.$$

These in turn can be used to determine c_6, c_7, c_8, etc. It is not too hard to find an explicit expression for the c's

$$\begin{aligned}
c_{3m} &= \frac{3m-2}{(3m)(3m-1)(3m-2)} c_{3(m-1)} \\
&= \frac{3m-2}{(3m)(3m-1)(3m-2)} \frac{3m-5}{(3m-3)(3m-4)(3m-5)} c_{3(m-1)} \\
&= \frac{(3m-2)(3m-5)\ldots 1}{(3m)!} c_0, \\
c_{3m+1} &= \frac{3m-1}{(3m+1)(3m)(3m-1)} c_{3(m-1)+1} \\
&= \frac{3m-1}{(3m+1)(3m)(3m-1)} \frac{3m-4}{(3m-2)(3m-3)(3m-4)} c_{3(m-2)+1} \\
&= \frac{(3m-2)(3m-5)\ldots 2}{(3m+1)!} c_1, \\
c_{3m+1} &= 0.
\end{aligned}$$

The general solution is thus

$$y(t) = a\left[1 + \sum_{m=1}^{\infty} c_{3m} t^{3m}\right] + b\left[1 + \sum_{m=1}^{\infty} c_{3m+1} t^{3m+1}\right].$$

The technique sketched here can be proven to work for any differential equation

$$y''(t) + p(t)y'(t) + q(t)y(t) = f(t)$$

provided that $p(t)$, $q(t)$ and $f(t)$ are analytic at $t = 0$. Thus if p, q and f have a power series expansion, so has y.

2 Singular points

Most equations of interest are of a form where p and/or q are *singular* at the point t_0 (usually $t_0 = 0$). Any point t_0 where $p(t)$ and $q(t)$ are singular is called a singular point. Of most interest are a special class of singular points *regular singular points*, where the differential equation can be given as

$$(t - t_0)^2 y''(t) + (t - t_0)\alpha(t)y'(t) + \beta(t)y(t) = 0,$$

with α and β analytic at $t = t_0$. Let us assume that this point is $t_0 = 0$. Frobenius' method consists of the following technique: In the equation

$$x^2 y''(x) + x\alpha(x)y'(x) + \beta(x)y(x) = 0,$$

we assume a generalised series solution of the form

$$y(x) = x^\gamma \sum_{n=0}^{\infty} c_n x^k.$$

Equating powers of x we find

$$\gamma(\gamma - 1)c_0 x^\gamma + \alpha_0 \gamma c_0 x^\gamma + \beta_0 c_0 x^\gamma = 0,$$

etc. The equation for the lowest power of x can be rewritten as

$$\gamma(\gamma - 1) + \alpha_0 \gamma + \beta_0 = 0.$$

This is called the indicial equation. It is a quadratic equation in γ, that usually has two (complex) roots. Let me call these γ_1, γ_2. If $\gamma_1 - \gamma_2$ is not integer one can prove that the two series solutions for y with these two values of γ are independent solutions.

3 Special cases

For the two special cases I will just give the solution (too much work to do it in all generality)

3.1 Two equal roots

If the indicial equation has two equal roots, $\gamma_1 = \gamma_2$, we have one solution of the form

$$y_1(t) = t^{\gamma_1} \sum_{n=0}^{\infty} c_n t^n.$$

The other solution takes the form

$$y_2(t) = y_1(t) \ln t + t^{\gamma_1 + 1} \sum_{n=0}^{\infty} d_n t^n.$$

Notice that this last solution is always singular at $t = 0$, whatever the value of γ_1!

3.2 Two roots differing by an integer

If the indicial equation that differ by an integer, $\gamma_1 - \gamma_2 = n > 0$, we have one solution of the form

$$y_1(t) = t^{\gamma_1} \sum_{n=0}^{\infty} c_n t^n.$$

The other solution takes the form

$$y_2(t) = a y_1(t) \ln t + t^{\gamma_2} \sum_{n=0}^{\infty} d_n t^n.$$

The constant a is determined by substitution, and in a few relevant cases is even 0, so that the solutions *can* be of the generalised series form.