2C1: P.D.E., Handout 8

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Bessel function and 2D problems

1 Bessel's equation

Bessel's equation of order ν is given by

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0.$$

$x = 0$ is a regular singular point. The indicial equation is

$$\alpha^2 - \nu^2 = 0$$

The generalised series solution gives two independent solutions if $\nu \neq \frac{1}{2}n$.

$$y = x^\nu \sum_n a_n x^n.$$

We find

$$\sum_n (n+\nu)(n+\nu-1))a_\nu x^{m+\nu} + \sum_n (n+\nu)a_\nu x^{m+\nu} + \sum_n (x^2-\nu^2)a_\nu = 0$$

which leads to

$$((n+\nu)^2 - \nu^2)a_n = -a_{n-2}$$

or

$$a_n = -\frac{1}{m(m+2\nu)}a_{n-2}.$$

If we take $\nu = n > 0$, we have

$$a_n = -\frac{1}{m(m+2n)}a_{n-2}.$$

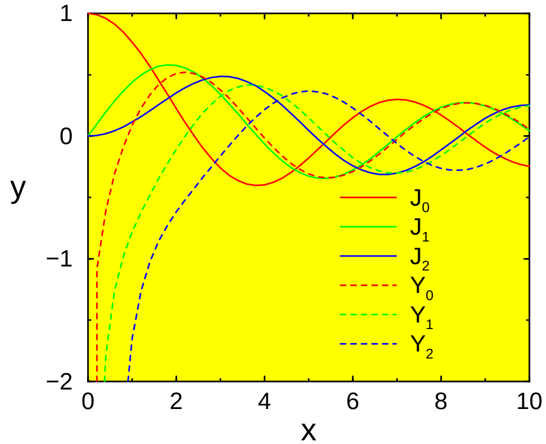
This can be solved by iteration,

$$\begin{aligned} a_{2k} &= -\frac{1}{4}\frac{1}{k(k+n)}a_{2(k-1)} \\ &= \left(\frac{1}{4}\right)^2 \frac{1}{k(k-1)(k+n)(k+n-1)}a_{2(n-2)} \\ &= \left(-\frac{1}{4}\right)^k \frac{n!}{k!(k+n)!}a_0. \end{aligned}$$

If we choose $a_0 = \frac{1}{n!2^n}$ we find the Bessel function of order n

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n}.$$

There is amother independent solution (which should have a logarithm in it) with goes to inifinity at $x = 0$.



The general solution is

$$y(x) = AJ_n(x) + BY_n(x),$$

where Y_n has a logarithmic singularity at the origin.

2 Properties of Bessel functions

Bessel functions have many interesting properties:

$$
\begin{aligned}
J_0(0) &= 1 &(1)\\
J_n(x) &= 0 \ \ (n>0) &(2)\\
J_{-n}(x) &= (-1)^n J_n(x) &(3)\\
\frac{d}{dx}\left[x^{-n}J_n(x)\right] &= -x^{-n}J_{n+1}(x) &(4)\\
\frac{d}{dx}\left[x^{n}J_n(x)\right] &= x^{n}J_{n-1}(x) &(5)\\
\frac{d}{dx}\left[J_n(x)\right] &= \tfrac{1}{2}\left[J_{n-1}(x)-J_{n+1}(x)\right] &(6)\\
xJ_{n+1}(x) &= 2nJ_n(x)-xJ_{n-1}(x) &(7)\\
\int [x^{-n}J_{n+1}(x)]\,dx &= -x^{-n}J_n(x)+C &(8)\\
\int [x^{n}J_{n-1}(x)]\,dx &= x^{n}J_n(x)+C &(9)
\end{aligned}
$$

3 Sturm-Liouville theory

We shall want to write a solution to an equation as a series of Bessel functions. We need to understand orthogonality of Bessel function. This is most easily done by developing a mathematical tool called Sturm-Liouville theory. It starts from an equation in the so-called self-adjoint form

$$[r(x)y'(x)]' + [p(x)+\lambda s(x)]y(x) = 0 \qquad (10)$$

where λ is a number, and $r(x)$ and $s(x)$ are greater than 0 on $[a,b]$. We apply the boundary conditions

$$
\begin{aligned}
a_1 y(a) + a_2 y'(a) &= 0,\\
b_1 y(b) + b_2 y'(b) &= 0,
\end{aligned}
$$

with a_1 and a_2 not both zero, and b_1 and b_2 similar.

Theorem 1. *If there is a solution to (10) then λ is real.*

Assume $\lambda = \alpha + i\beta$, with solution Φ. By complex conjugation find

$$
\begin{aligned}
[r(x)\Phi'(x)]' + [p(x)+\lambda s(x)]\Phi(x) &= 0\\
[r(x)(\Phi^*)'(x)]' + [p(x)+\lambda^* s(x)](\Phi^*)(x) &= 0
\end{aligned}
$$

where $*$ denotes complex conjugation. Multiply the first equation by $\Phi^*(x)$ and the second by $\Phi(x)$, and subtract the two equations, integrate over x from a to b and find

$$(\lambda^*-\lambda)\int_a^b s(x)\Phi^*(x)\Phi(x)\,dx = \int_a^b \Phi(x)[r(x)(\Phi^*)'(x)]' - \Phi^*(x)[r(x)\Phi'(x)]'\,dx$$

The second part can be integrated by parts, and we find

$$
\begin{aligned}
&(\lambda^*-\lambda)\int_a^b s(x)\Phi^*(x)\Phi(x)\,dx\\
&\quad = r(b)\left[\Phi'(b)(\Phi^*)'(b) - \Phi^*(b)\Phi'(b)\right] - r(a)\left[\Phi'(a)(\Phi^*)'(a) - \Phi^*(a)\Phi'(a)\right] = 0,
\end{aligned}
$$

where the last step can be done using the boundary conditions. Since both $\Phi^*(x)\Phi(x)$ and $s(x)$ are greater than zero we conclude that $\int_a^b s(x)\Phi^*(x)\Phi(x)\,dx > 0$, which can now be divided out of the equation to lead to $\lambda = \lambda^*$.

Theorem 2. *Let Φ_n and Φ_m be two solutions for different values of λ, $\lambda_n \neq \lambda_m$, then*

$$\int_a^b s(x)\Phi_n(x)\Phi_m(x)\,dx = 0.$$

The proof is to a large extend identical to the one above: multiply the equation for $\Phi_n(x)$ by $\Phi_m(x)$ and vice-versa. Subtract and find

$$(\lambda_n-\lambda_m)\int_a^b s(x)\Phi^m(x)\Phi_n(x)\,dx = 0$$

which leads us to conclude that

$$\int_a^b s(x)\Phi_n(x)\Phi_m(x)\,dx = 0.$$

Theorem 3. *Under the conditions set out above*
a) There exists a real infinite set of eigenvalues $\lambda_0,\ldots,\lambda_n,\ldots$ with $\lim_{n\to\infty} = \infty$.
b)If Φ_n is the eigenfunction corresponding to λ_n, it has exactly n zeroes in $[a,b]$.

Clearly the Bessel equation is of self-adjoint form: rewrite

$$x^2y'' + xy' + (x^2-n^2)y = 0$$

as (divide by x)

$$[xy']' + (x - \frac{n^2}{x})y = 0$$

We *cannot* identify n with λ, and we *do not* have positive weight functions. It can be proven from properties of the equation that the Bessel functions have an infinite number of zeroes on the interval $[0, \infty)$. A small list of these:

J_0	:	2.42	5.52	8.65	11.79	...
$J_{1/2}$	:	π	2π	3π	4π	...
J_8	:	11.20	16.04	19.60	22.90	...

4 Temperature on a disk

A circular disk is prepared such that the initial temperature is

$$u(\rho, \phi, t = 0) = f(\rho).$$

Then it is placed between two pefect insulators and its circumference is connected to a freezer that keeps it at 0° C, as sketched in Fig. 1.

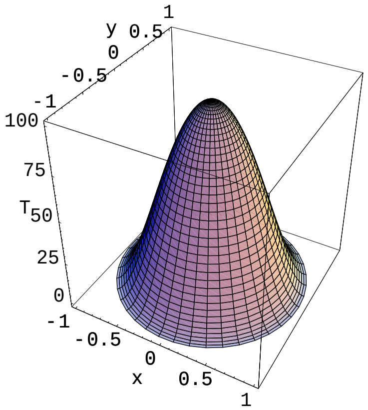


Figure 1: the initial temperature in the disk

Since the initial conditions do not depend on ϕ, we expect the solution to be radially symmetric as well, $u(\rho, t)$,

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\left[\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho}\right], \\ u(c, t) &= 0, \\ u(\rho, 0) = f(\rho). \end{aligned}$$

Separate variables, $u(\rho, t) = R(\rho)T(t)$,

$$\frac{1}{k}\frac{T'}{T} = \frac{R'' + \frac{1}{\rho}R'}{R} = -\lambda$$

$$\begin{aligned} \rho^2 R'' + \rho R' + \lambda\rho^2 R &= 0 \quad R(c) = 0 \\ T' + \lambda k T = 0. \end{aligned}$$

Change variables to $x = \sqrt{\lambda}\rho$. We find

$$\frac{d}{d\rho} = \sqrt{\lambda}\frac{d}{dx},$$

and we can remove a common factor $\sqrt{\lambda}$ to obtain $(X(x) = R(\rho))$

$$[xX']' + xX = 0,$$

which is Bessel's equation of order 0, i.e.,

$$R(\rho) = J_0(\rho\sqrt{\lambda}).$$

The boundary condition $R(c) = 0$ shows that

$$c\sqrt{\lambda_n} = x_n,$$

where x_n are the zero points of J_0. We thus conclude

$$R_n(\rho) = J_0(\rho\sqrt{\lambda_n}).$$

the first five solutions R_n (for $c = 1$) are shown in Fig. 2.

From Sturm-Liouville theory we conclude that

$$\int_0^\infty \rho d\rho\, R_n(\rho) R_m(\rho) = 0 \text{ if } n \neq m.$$

Together with the solution for the T equation,

$$T_n(t) = \exp(-\lambda_n k t)$$

we find a Fourier-Bessel series type solution

$$u(\rho, t) = \sum_{n=1}^{\infty} A_n J_0(\rho\sqrt{\lambda_n}) \exp(-\lambda_n k t),$$

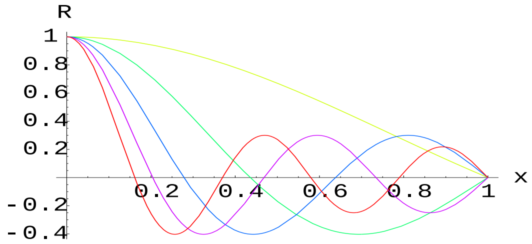


Figure 2: A graph of the first five functions R_n

with $\lambda_n = \alpha_n^2 = (x_n/c)^2$.

So for our initial problem we would have to calulate the integrals (see the next section for an explanation)

$$A_j = \frac{2}{c^2 J_1^2(c\alpha_j)} \int_0^c f(\rho) J_0(\alpha_j \rho) \rho d\rho.$$

5 Fourier-Bessel series

So how can we determine in general the coefficients in a Fourier-Bessel series

$$f(\rho) = \sum_{j=1}^{\infty} J_n(\alpha_j \rho)?$$

We impose $f(c) = J_n(\alpha_j c) = 0$, for ease of calculation. Other boundary conditions can be dealt with in the same way.

Write $R_j = J_n(\alpha_j \rho)$,

$$(\rho R_j')' + (\alpha_j^2 \rho - \frac{n^2}{\rho}) R_j = 0.$$

Where we assume that f and R satisfy the boundary condition

$$\begin{aligned} f(c) &= \; = 0, \\ R_j(c) &= 0 \end{aligned}$$

From Sturm-Liouville theory we do know that

$$\int_0^c \rho J_n(\alpha_i \rho) J_n(\alpha_j \rho) = 0 \text{ if } i \neq j,$$

but we shall also need the values when $i = j$!

Use Bessel's equation, multiply with $2\rho R'$, and integrate over ρ from 0 to c,

$$\int_0^c \left[(\rho R_j')' + (\alpha_j^2 \rho - \frac{n^2}{\rho}) R_j \right] 2\rho R_j' d\rho = 0$$

We find

$$\begin{aligned} \int_0^c \frac{d}{d\rho} \left(\rho R_j'\right)^2 d\rho &= 2n^2 \int_0^c R_j R_j' d\rho - 2\alpha_j^2 \int_0^c \rho^2 R_j R_j' d\rho \\ \left(\rho R_j'\right)^2 \Big|_0^c &= n^2 R_j^2 \big|_0^c - 2\alpha_j^2 \int_0^c \rho^2 R_j R_j' d\rho \end{aligned}$$

The last integral can be done by parts:

$$2 \int_0^c \rho^2 R_j R_j' d\rho = \int_0^c \rho^2 (R_j^2)' d\rho = -2 \int_0^c \rho^2 R_j^2 d\rho + \rho^2 R_j^2 \big|_0^c$$

So we finally conclude that

$$2\alpha_j^2 \int_0^c \rho R_j^2 d\rho = \left[\left(\alpha_j^2 \rho^2 - n^2\right) R_j^2 + \left(\rho R_j'\right)^2 \right|_0^c .$$

Use the boundary conditions $R_j(c) = 0$, and find

$$R_j' = \alpha_j J_n'(\alpha_j \rho).$$

We conclude that

$$2\alpha_j^2 \int_0^c \rho R_j^2 d\rho = \left[\left(\rho \alpha_j J_n'(\alpha_j \rho)\right)^2 \right|_0^c = c^2 \alpha_j^2 \left(J_{n+1}(\alpha_j c)\right)^2$$

We thus finally have our result

$$\int_0^c \rho R_j^2 d\rho = \frac{c^2}{2} J_{n+1}^2(\alpha_j c).$$

Why good we have guessed the presence of the factor ρ?

Example 1:

Consider the function

$$f(x) = \begin{cases} x^3 & 0 < x < 10 \\ 0 & x > 10 \end{cases}$$

Expand this function in a Fourier-Bessel series using J_3.

Solution:

From our definitions we find that

$$f(x) = \sum_{j=1}^{\infty} A_j J_3(\alpha_j x),$$

with

$$\begin{aligned} A_j &= \frac{2}{100 J_4(10\alpha_j)^2} \int_0^{10} x^3 J_3(\alpha_j x) dx \\ &= \frac{2}{100 J_4(10\alpha_j)^2} \frac{1}{\alpha_j^5} \int_0^{10\alpha_j} s^4 J_3(s) ds \\ &= \frac{2}{100 J_4(10\alpha_j)^2} \frac{1}{\alpha_j^5} (10\alpha_j)^4 J_4(10\alpha_j) ds \\ &= \frac{200}{\alpha_j J_4(10\alpha_j)} \end{aligned}$$

The first five values of A_j are $1050.95, -821.503, 703.991, -627.577, 572.301$, and the first five partial sums are plotted in Fig. 3.

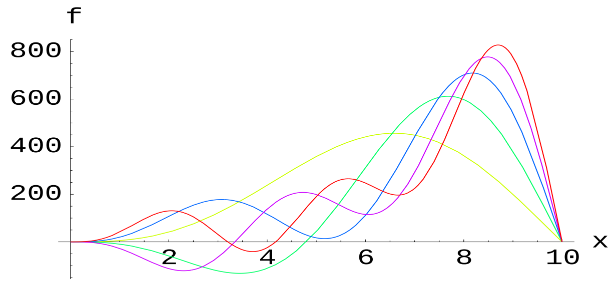


Figure 3: A graph of the first five partial sums for x^3 expressed in J_3.