GENERAL RELATIVITY: WHY?

The subject of this course is Einstein's Theory of GENERAL RELATIVITY:

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Einstein theory of gravity:

(1) is crucial to understand phenomena at the frontiere of astrophysics, such as black-holes, pulsar, quasars

(2) is crucial for the study of the origin and evolution of the universe: cosmology.

(3) BUT nowadays is also crucial in every day life: GPS wouldn't work without the computation of relativistic effects on the orbits of planets (and satellites)!

General relativity was born after 1905.

BEFORE 1905: Newtonian gravity was the commonly accepted description of gravitational laws.

According to the law of Newton the gravitational force between two bodies of masses m and M and far apart at a quantity r_{12} had magnitude

$$F = G \frac{m M}{r_{12}^2} \qquad (1)$$

where the Newton gravitational const. $G = 6{,}67 \times 10^{-11}\ m^3/Kg\,s^2$

IMPORTANT #1: GRAVITY IS UNIVERSAL

The force $\vec{F}$ is directed along a line connecting the two bodies and it is ALWAYS attractive. | KEY FACT # 2: GRAVITY IS UNSCREENED

Notice that the force above can be interpreted as the force exerted by a point of mass M in x_M on a point of mass m, located in $\vec{x}$ such that $|\vec{x}_M - \vec{x}| = r$, and that it can be rewritten as:

$$\boxed{\vec{F} = -m \, \vec{\nabla}\Phi(\vec{x})}$$

where $\Phi(\vec{x})$ is the gravitational potential generated by the mass M. Comparing with ①

$$\Phi(\vec{x}) = -\frac{GM}{r} = -\frac{GM}{|x_M - x|}$$

for a discrete distribution of masses M_i placed in x_i

$$\Phi(\vec{x}) = -\sum_i G\frac{M_i}{|x - x_i|}$$

continuous limit : distribution of density $\rho(x)$

$$\Phi(x) = -G\int d^3x' \, \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} \longrightarrow \text{NB}: dm = \rho(x')\, d^3x'$$

This is basically a law for the gravitational potential, which can be put in differential form considering the flux of $\vec{\nabla}\Phi$ through a closed surface and we apply the Gauss Theorem:

POISSON EQUATION

$$\boxed{\nabla^2\Phi(\vec{x}) = -4\pi G\rho(\vec{x})}$$

EXERCISE : Try to integrate this BACK to get

$$\Phi = -G\frac{M}{r}$$

DEFINING EQUATION FOR THE NEWTONIAN

WHAT DID IT HAPPEN IN 1905? which made people (well scientists) realize that something was missing in the description of gravity?

$\boxed{\text{EINSTEIN FORMULATED SPECIAL RELATIVITY}}$ and this was the origin point of a conceptual revolution which lead, together with the birth of quantum mechanics, to modern physics

One of the key assumption of S.R. is
POSTULATES

FINITNESS OF THE SPEED OF LIGHT: and it is the same in any inertial reference frame.

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NO INSTANT TRANSMISSION OF INFORMATION!

no information can be transmitted instantaneously.

BUT Newton's laws are instantaneous!
They describe an instantaneous interaction between the two masses m and M, (if I change M, this affects instantaneously the force felt by m), thus

$\boxed{\text{NEWTON'S LAW IS INCOMPATIBLE WITH SPECIAL RELATIVITY AND WITH THE FINITNESS OF THE SPEED OF LIGHT.}}$

Furthermore, $\boxed{\text{SPECIAL RELATIVITY DEALS WITH INERTIAL FRAMES ONLY}}$ so, by itself, it is incomplete

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There must exist a more general theory, a truly fundamental theory of gravitation [1], which describes the behaviour of accelerated frames in a way which is compatible with S.R, and such that Newtonian gravity is a well defined limit of it. [2]

This theory was formulated by Einstein himself in 1915, and it lead to a new, even more profound conceptual revolution:

Already special relativity showed that space and time are not disconnected objects but form a unique substrate where to formulate naturally the laws of physics. [SPACETIME] which can rotate the one into the other!

The observation that all the objects moving in a gravitational field are subjected to the same acceleration lead Einstein to UNDERSTAND GRAVITY AS THE CURVATURE OF 4 dimensional spacetime

⇒ MASSES CURVE SPACETIME IN ITS VICINITY

⇒ OTHER MASSES NEARBY FREELY FALL ALONG TRAJECTORIES WHICH ARE THE "STRAIGHT PATHS" IN THIS CURVED GEOMETRY

⇓

KEY FACT #3: GRAVITY IS GEOMETRY.

This is summed up nicely in the Einstein equations:

$$\underbrace{G_{\mu\nu}}_{\text{geometry of spacetime}} \propto \underbrace{T_{\mu\nu}}_{\text{matter contents (energy + matter!) of spacetime}}$$

Notice furthermore that:

EXT. OF ④

KEY FACT # 4: GRAVITY IS UNIVERSAL : it is the interaction between all masses (and energies!)
$E = mc^2$.

KEY FACT #5: GRAVITY IS WEAK

is the weakest of the fundamental interaction acting between elementary particles at an accessible scale

For two protons separated by a distance r

$$\left.\begin{aligned} F_{grav} &= G \frac{m_p^2}{r^2} \\ F_{el} &= \frac{1}{4\pi\epsilon_0} \frac{e^2}{r^2} \end{aligned}\right\} \frac{F_{grav}}{F_{el}} = 4\pi\epsilon_0 G \frac{m_p^2}{e^2} \sim 10^{-36}$$

KEY FACT # 6: GRAVITY IS A LONG RANGE INTERACTION

Unlike weak and strong interaction, there is not a fundamental scale which sets up a finite range of interaction, thus $F \propto \frac{1}{r^2}$ with $r \to \infty$

$\Rightarrow$ UNSCREENED AND LONG RANGE. despite being the weakest it governs the organization of the universe at big scales!

WHEN DOES GENERAL RELATIVITY BECOMES RELEVANT?

There exists a characteristic dimensionless ratio which determines when the relativistic aspects of gravity becomes important:

$$\frac{GM}{Rc^2}$$

$\Rightarrow$ if this is a considerable fraction of unit then G.R. is crucial to describe the physics of the investigated object!

Resembles v^2/c^2 for special Rel:

$\rightarrow$ ON THE SURFACE OF EARTH:

$$\frac{GM_\oplus}{c^2 R_\oplus} \sim 10^{-9}$$

$\rightarrow$ ON THE SURFACE OF THE SUN

$$\frac{GM_\odot}{c^2 R_\odot} \sim 10^{-6}$$

NOT very RELATIVISTIC OBJECTS

$\Downarrow$

HOWEVER, RELATIVISTIC EFFECTS ARE THERE and it is possible to quantify them precisely:

$\Rightarrow$ GPS: tiny effects of G.R. are important

$\Rightarrow$ MERCURY PERIHELIUM position shifts every year of a tiny but detectable amount: this is a truly relativistic effect.

→ ON THE SURFACE OF A NEUTRON STAR $(M \sim M_\odot,\ R \sim 10\,km)$

$$\frac{GM}{c^2R} \sim 0{,}1 \Rightarrow \text{relativistic object.}$$

→ BLACK HOLE : escape velocity > c if

$$\frac{2GM}{c^2R} > 1$$

this is similar to the (Newtonian) criterion above.

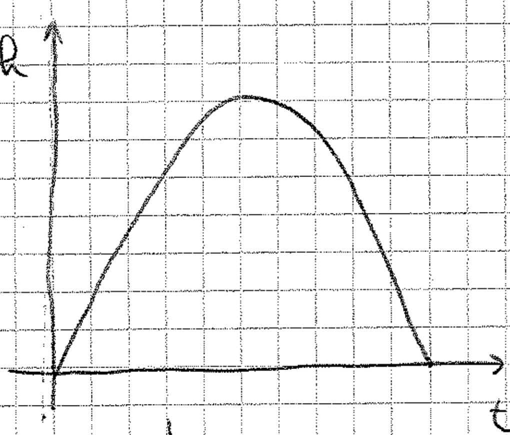
GRAVITY AS GEOMETRY

Experiments have shown that

ALL THE BODIES FALL WITH THE SAME ACCELERATION.

You can try, and throw different objects in the air, and measure the height they reach as function of time.

In every case you get the same curve, the same parabola, whose slope DOES NOT DEPEND at all on the mass of the object which has been thrown up in the air.



$\Downarrow$

FIRST HINT THAT GRAVITY IS AN INTRINSIC PROPERTY OF SPACETIME RATHER THAN AN EFFECT DUE TO A PROPAGATING FIELD THAT EXERTS A FORCE ON SURROUNDING OBJECTS.

This universality of gravity has been rigorously formulated in the so-called

EQUIVALENCE PRINCIPLE

The equivalence principle comes in many forms, the first of which states that:

WEAK EQUIVALENCE PRINCIPLE

↓ states that

THE INERTIAL MASS AND THE GRAVITATIONAL MASS OF A BODY DO COINCIDE.

To understand what this does mean, let's start from the second Newton's law:

$$\vec{F} = m_i \vec{a}$$

it says that the force exerted on an object is proportional to the acceleration it undergoes.

The constant of proportionality is the INERTIAL mass of the object.

I.M. As the same name says, this mass is a measure of the inertia of the object, of the resistance that we feel when we try to push the object.

Thus, this notion of mass is UNIVERSAL, the Newton law hold whichever force we are considering.

If we consider the laws of gravity, things apparently go in a different way. If we have an object moving in a gravitational field, we know that the force that this body feel is proportional to the gradient of the potential generating the field through the relation

$$\vec{F} = -m_g \vec{\nabla \phi} \xRightarrow{\text{know}} F = G\frac{mM}{r^2}$$ (10)

where the constant of proportionality, m_g, is the GRAVITATIONAL MASS OF THE OBJECT.

Conceptually, the gravitational mass is something quite different from the inertial mass, and it is something uniquely related to the gravitational force that we are considering.

HOWEVER: experiments show that the response of matter to gravitation is universal:

if you drop two very different object, they fall with the same acceleration, independently from the composition of the object. →

↓

THERE IS NO "GRAVITATIONAL CHARGE" WHICH CHANGE HOW AN OBJECT FEELS THE ACTION OF A GRAVITATIONAL FIELD!

This means that a body respond to gravity exactly as it responds to any other force, and

$$\boxed{m_i \equiv m_g}$$

→ mention the pendulum!

$$T = 2\pi\sqrt{\frac{l}{g}\frac{m_I}{m_G}}$$

But if we observe the formulas that we just wrote, this has a very important consequence:

$$\boxed{\vec{a} = - \vec{\nabla}\Phi}$$

which means that the acceleration felt by a body which moves freely in a gravitational field is UNIVERSAL:

1. : IT IS AN INTRINSIC PROPERTY OF THE FIELD

2. INDEPENDENT ON ANY CHARACTERISTIC OF THE TEST PARTICLE (e.g. its mass)

(in the field)

This ~~postulating~~ suggests that it exists a preferred set of trajectories called

INERTIAL or FREELY FALLING

along which travel UNACCELERATED PARTICLES

Where unaccelerated particles means: particle subjected only to gravity.

Notice that these are perfectly equivalent to the inertial reference frames of classical and rel. mechanics.

~~Equivalent~~

To understand this point better you can think about what Einstein thought, actually about what he defined as "the happiest thought of all my life"

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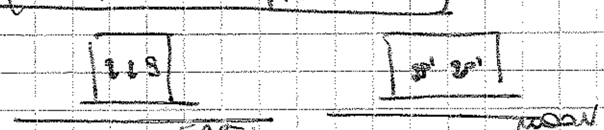
THE GRAVITATIONAL FIELD HAS ONLY A RELATIVE EXISTANCE!

⇒ the thought of a person falling freely from the roof of a house ⇒ he notice that for such a person
THERE IS NO GRAVITATIONAL FIELD AROUND.

To understand better the consequences of his thought, let's consider a "less cruent" situation where we can analyze what's happening without waiting of somebody smashing on the ground!

This is a quite canonical experiment, where we have a scientist which is tightly sealed in a box and he cannot see anything of the outside.

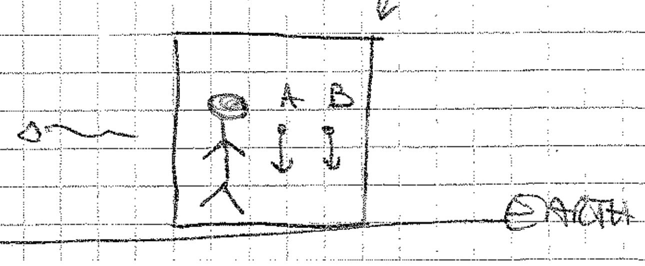
Imagine this person is doing experiments and he is trying to investigate the properties of the gravitational field by dropping two different objects.



HE FINDS
Then the acceleration he would get would be ≠ moving the box somewhere else, eg. the moon, or empty space.
2

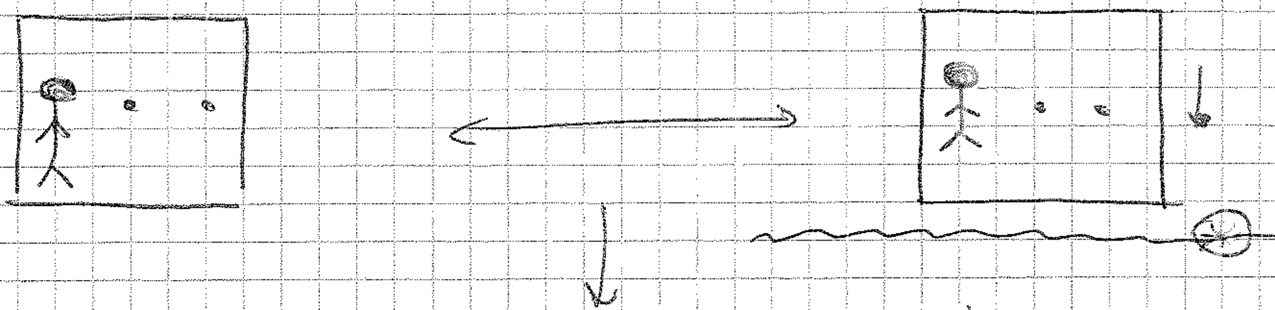
However, let's compare these two situations:

1 finds object fall with the same acceleration



BOX IN EMPTY SPACE
(far from any gravitational source)

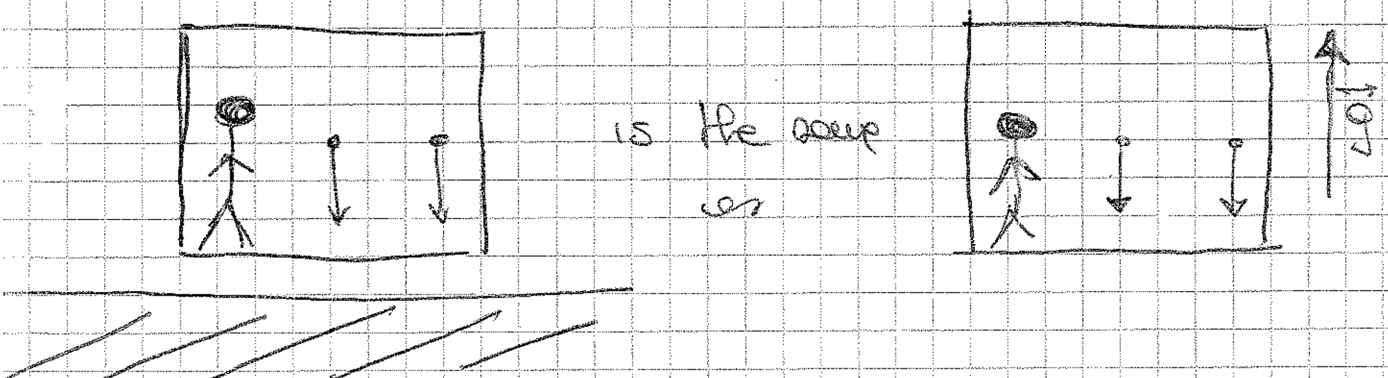
BOX FREELY FALLING



can he distinguish between these two situations?

NO! everything is "weightless" in both cases. (*) there "is no gravitational field" for the person inside the box. (**)

This actually mean that we can also FAKE a gravitational field! We just have to accelerate a box in space



(**) Notice that $m_i = m_g$ is crucial for this experiment. If two different objects fall toward Earth with different acceleration, then they would not be at rest one respect to the other in (*), and so the presence of the gravitational field would be detected.

This is totally different from what happen in EM: In this case particle with different "charge" would react differently to the electromagnetic field. Their acceleration depends on the charge!

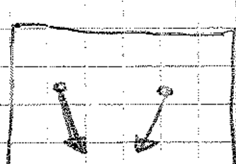
Here the "gravitational charge", which we can think as the ratio $\boxed{M_g/M_i}$ is the same (1) for all objects.

However, notice $\boxed{\text{that we must be careful}}$. We can actually apply the arguments above only if we are in small regions of spacetime!

If the box were big, then the gravitational field would be different from one point to another, and these differences would be detectable and couldn't be just "faked" with a uniform acceleration. (*)

After all these considerations, we can rephrase ~~the notion~~ the WEAK EQUIVALENCE PRINCIPLE AS:

THE NOTIONS OF FREELY-FALLING PARTICLES ARE THE SAME IN A GRAVITATIONAL FIELD AND IN A UNIFORMLY ACCELERATED FRAME IN SMALL ENOUGH REGION OF SPACE

(*) 

⇝ these inhomogeneities lead to tidal forces.

HOWEVER, after 1905, after special relativity, the concept of mass lost some of its importance: mass was just another manifestation of energy.

Then the laws of physics changed, the relativistic regime was introduced, so Einstein thought of a generalization of the WEP which would include naturally all these new developments. The idea was simply that for the physicist in the freely falling box would be impossible to distinguish between the external grav. field and an uniform acceleration, HE WOULD NOT FEEL ANY ACCELERATION NO MATTER WITCH KIND OF EXPERIMENTS HE WAS DOING.

This was formalized with the → the laws of physics he could observe were those of special relativity!

EINSTEIN EQUIVALENCE PRINCIPLE

IN SMALL ENOUGH REGIONS OF SPACETIME, THE LAWS OF PHYSICS REDUCES TO THOSE OF SPECIAL RELATIVITY; IT IS IMPOSSIBLE TO DETECT THE EXISTANCE OF A GRAVITATIONAL FIELD BY MEANS OF LOCAL EXPERIMENTS

↓

Ultimately, the EEP implies that we should attribute the effects of gravity to the curvature of spacetime = curvature of trajectories that freely falling objects follow

This is the profound insight of general relativity.
Gravity is not due to a propagating field, but masses curve spacetime, and the bodies freely fall in ~~these~~ curved space, were "free-fall" means that they follow ~~s~~the equivalent of straight lines in this curved space.

Thus gravity is due to the geometry of spacetime. (16)
The quantitative description (formal description) of this concept will come with the EINSTEIN EQUATIONS

$$G_{\mu\nu} \propto T_{\mu\nu}$$

(geometry of spacetime) = (matter content of spacetime)

Furthermore, since gravity is inexplicable from an acceleration, then the concept of "acceleration" due to gravity is of little use.

For this, <u>the notion of inertial frame is replaced by that of FREELY FALLING FRAME</u>

frame which is subjected only to the LOCAL effects of gravity; it which is moving freely in a gravitational field; ie in the proximity of an object deforming spacetime.

THIS HAS TO BE "SMALL" AND LOCALIZED IN "SMALL" REGION OF SPACETIME

NOTICE THAT due to this it is no longer possible built a "UNIQUE" inertial (freely falling) frame due to the inhomogeneities of ~~the~~ gravitational field.

~~A inertial frame~~

If we are in a freely falling frame and we build our reference frame as a rigid structure of rods, at some large distance a freely-falling object will look as it is accelerating with respect to this frame.

Thus, the solution is to retain the notion of inertial frame, but forget about extending it uniquely throughout spacetime.

OUR SUBJECTS WILL THEN BE

LOCALLY INERTIAL FRAMES (*): ie those following the motion of freely falling particles.

and the EEP can be rephrased by saying

AT ANY POINT IN SPACETIME, IN ANY ARBITRARY GRAVITATIONAL FIELD, IT IS POSSIBLE TO CHOSE A LOCALLY INERTIAL FRAME SUCH THAT IN AN ARBITRARY NEIGHBORHOOD OF THAT POINT THE LAWS OF PHYSICS ASSUME THE FORM AS IN AN UNACCELERATED COORDINATE SYSTEM IN THE ABSENCE OF GRAVITY.

(*) BUT WHICH IS SMALL ENOUGH TO MAKE THE INHOMOGENITIES OF THE GRAVITATIONAL FIELD UNDETECTABLE!

TIDAL FORCES IN THE NEWTON THEORY

We have seen that

→ masses curve spacetime, and gravity is nothing but the effect of this curvature

→ object follow the equivalent of straight lines in this curved geometry.

Before launching ourselves in the study of curved geometry, we should wonder if there is a way to find a measure of this spacetime curvature already in Newtonian gravity.

→ this can be done by observing the relative motion of two test particles in the proximity of the earth. This would provide a mathematical description of non-local effects.

So let's consider two particles in $\vec{x}$ and $\vec{y}$, which are a quantity $\vec{\xi}$ apart: $\vec{y} = \vec{x} + \vec{\xi}$

In general in a gravitational field $\ddot{\vec{x}} = \vec{a} = -\vec{\nabla}\Phi(\vec{x})$, so

$$\ddot{\vec{y}} = \ddot{\vec{x}} + \ddot{\vec{\xi}}$$

and $\ddot{\vec{y}} = -\vec{\nabla}\Phi(\vec{y})$ ⇒ in components:

$$\ddot{y}^i = -\frac{\partial\Phi}{\partial y^i}$$

Now let's perform a change of variables $\vec{y} \to \vec{x} = \vec{y} - \vec{\xi}$

$$\ddot{y}^i = \ddot{x}^i + \ddot{\xi}^i = -\frac{\partial}{\partial x^i}\phi(\vec{x}+\vec{\xi})$$

and we can assume $\vec{x}^i$ and $\vec{y}^i$ are infinitesimally small, so we can expand for ξ^i small

$$-\frac{\partial}{\partial x^i}\phi(\vec{x}^i+\vec{\xi}^i) \simeq -\frac{\partial}{\partial x^i}\left(\phi(\vec{x}) + \sum_j \frac{\partial \phi}{\partial x^j}\xi^j\right)$$

$$\simeq \underbrace{-\frac{\partial \phi(\vec{x})}{\partial \vec{x}^i}}_{\equiv \ddot{x}^i} + \sum_j \frac{\partial^2 \phi}{\partial x^i \partial x^j}\xi^j$$

and we get

$$\cancel{\ddot{x}^i} + \ddot{\xi}^i \simeq \cancel{\ddot{x}^i} + \sum_j \frac{\partial^2 \phi(x)}{\partial x^i \partial x^j}\xi^j$$

and if we call

$$K_{ij} \equiv \frac{\partial^2 \phi(x)}{\partial x^i \partial x^j}$$

then

$$\ddot{\xi}^i \simeq K_{ij}\xi^j$$

THESE ARE THE COMPONENTS OF THE RELATIVE ACCELERATION BETWEEN THE TWO PARTICLES.

<u>NB:</u> $\mathrm{Tr}\,K = \nabla^2\phi = 4\pi G \rho_M$