(45)

WICK'S THEOREM

FROM DYSON'S FORMULA WE WANT TO COMPUTE ~~THINGS~~

$$\langle f|S|i\rangle = \langle f|\, T\, e^{-i\int_{-\infty}^{+\infty} H_I(t')\,dt'} \,|i\rangle$$

WHERE $H_I = \int d^3x\, \mathcal{H}_I = -\int d^3x\, \mathcal{L}_I$

$\uparrow$ INTERACTION PART OF LAGRANGIAN

(OR IF WE EXPAND DYSON'S FORMULA WE WANT TO EVALUATE THINGS LIKE

$$(-i)^n \langle f|\, T\{H_I(x_1) \ldots H_I(x_n)\}\,|i\rangle$$

$$= \langle 0|\, a^+(\vec{p}_{m+n}) \ldots a^+(\vec{p}_{m+1}) \qquad\qquad = a^+(\vec{p}_m)\, a^+(\vec{p}_{m-1}) \ldots a^+(\vec{p}_1)\,|0\rangle$$

AN EASY EXAMPLE: CONSIDER THE REAL SCALAR FIELD

$$\phi(x) = \underbrace{\int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}} \left(a(\vec{p})\, e^{-ip\cdot x}\right)}_{\equiv\, \phi^{(+)}(x)} + \underbrace{\int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}}\, a^+(\vec{p})\, e^{ip\cdot x}}_{\equiv\, \phi^{(-)}(x)}$$

(

CONSIDER THE FEYNMAN PROPAGATOR

$$\begin{aligned}
x^0 > y^0:\quad T\,\phi(x)\,\phi(y) &= \left(\phi^{(+)}(x) + \phi^{(-)}(x)\right)\left(\phi^{(+)}(y) + \phi^{(-)}(y)\right) \\
&= \phi^{(+)}(x)\,\phi^{(+)}(y) + \phi^{(-)}(x)\,\phi^{(+)}(y) \\
&\quad + \phi^{(-)}(x)\,\phi^{(+)}(y) + [\phi^{(+)}(x), \phi^{(-)}(y)] + \phi^{(-)}(x)\,\phi^{(-)}(y) \\
&= :\phi(x)\,\phi(y): + [\phi^{(+)}(x), \phi^{(-)}(y)]
\end{aligned}$$

$y^0 > x^0$:

$$T\,\phi(x)\,\phi(y) = \phi(y)\,\phi(x) = \ldots = :\phi(x)\,\phi(y): + [\phi^{(+)}(y), \phi^{(-)}(x)]$$

$$\Rightarrow T\,\phi(x)\,\phi(y) = \theta(x^0-y^0)\,\phi(x)\,\phi(y) + \theta(y^0-x^0)\,\phi(y)\,\phi(x)$$

$\downarrow$ OPERATOR

$$= :\phi(x)\,\phi(y): + \theta(x^0-y^0)\underbrace{[\phi^{(+)}(x), \phi^{(-)}(y)]}_{iD(x-y)} + \theta(y^0-x^0)\underbrace{[\phi^{(+)}(y), \phi^{(-)}(x)]}_{iD(y-x)}$$

$$= \underbrace{:\phi(x)\,\phi(y):}_{\text{OPERATOR}} + \underbrace{i\Delta_F(x-y)}_{\text{Function}}$$

- WICK'S THEOREM CONVERTS ~~PRODUCTS OF~~ TIME ORDERED PRODUCTS OF OPERATORS INTO PRODUCTS OF NORMAL ORDERED PRODUCTS AND FEYNMAN PROPAGATORS (RECALL $\langle 0|$ (NORMAL ORDERED PRODUCT) $|0\rangle = 0$ ALWAYS!)

<u>DEFINITION</u> CONTRACTION OF A PAIR OF FIELDS IN A STRING OF OPERATORS $\ldots\phi(x_1)\ldots\phi(x_2)\ldots$ MEANS TO REPLACE THOSE OPERATORS BY THEIR ~~[illegible]~~ FEYNMAN PROPAGATOR

$$(\ldots\underbrace{\phi_u(x_1)\;\ldots\;\phi(x_2)}\ldots) = i\Delta_F(x_1-x_2)\times(\ldots\ldots)$$

EX1) $\underbrace{\phi(x)\,\phi(y)} = i\Delta_F(x-y)$ <u>REAL SCALAR FIELD</u>

EX2) $\underbrace{\phi(x)\,\phi^\dagger(y)} = i\Delta_F(x-y)$ <u>COMPLEX SCALAR FIELD</u>

WHILE $\underbrace{\phi(x)\,\phi(y)} = \underbrace{\phi^\dagger(x)\,\phi^\dagger(y)} = 0$

WICK'S THEOREM (WITHOUT PROOF)

FOR ANY COLLECTION OF FIELDS $\phi_i \equiv \phi(x_i)$

$$T(\phi_1 \ldots \phi_n) = :\phi_1 \ldots \phi_n: + :\text{ALL POSSIBLE CONTRACTIONS}:$$

$$T\,\phi_1 \phi_2 = :\phi_1 \phi_2: + \underbrace{\phi_1 \phi_2}$$

$$T(\phi_1 \phi_2 \phi_3 \phi_4) = :\phi_1 \phi_2 \phi_3 \phi_4:$$

$$+ \underbrace{\phi_1 \phi_2} :\phi_3 \phi_4: + \underbrace{\phi_1 \phi_3} :\phi_2 \phi_4: + 4 \text{ TERMS}$$

$$+ \underbrace{\phi_1 \phi_2}\,\underbrace{\phi_3 \phi_4} + \underbrace{\phi_1 \phi_3}\,\underbrace{\phi_2 \phi_4} + \underbrace{\phi_1 \phi_4}\,\underbrace{\phi_2 \phi_3}$$

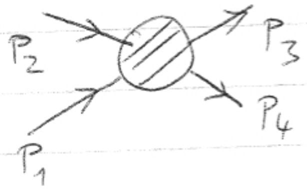
COMMENT IF WE CALCULATE $\langle 0 | T(\phi_1 \ldots \phi_n) | 0 \rangle$ ONLY CONTRACTIONS SURVIVE! ALSO, IF n IS ODD THEN $\langle 0 | T\,\phi_1 \ldots \phi_n | 0 \rangle = 0$!

LOWEST ORDER CALCULATIONS OF SCATTERING AMPLITUDES IN ϕ^4- THEORY

USE: $\mathcal{L}_I = -i\lambda \frac{1}{4!} :\phi^4(x):$

$$S = 1 + \frac{(-i\lambda)}{4!}\int d^4x :\phi^4(x): + \frac{1}{2}\frac{(-i\lambda)^2}{(4!)^2}\int d^4x\, d^4y\; T :\phi^4(x)::\phi^4(y): + \mathcal{O}(\lambda^3)$$

CONSIDER THE SCATTERING OF 2 PARTICLES INTO 2 PARTICLES



$$\langle \vec{p}_3, \vec{p}_4 | S | \vec{p}_1, \vec{p}_2 \rangle = ?$$

0^{TH}-ORDER: $\langle \vec{p}_3, \vec{p}_4 | \vec{p}_1, \vec{p}_2 \rangle = \langle 0 | a(\vec{p}_3)\, a(\vec{p}_4)\, a^\dagger(\vec{p}_1)\, a^\dagger(\vec{p}_2) | 0 \rangle$

SEE HOMEWORK

1^{ST} ORDER: $\frac{(-i\lambda)}{4!}\int d^4x \; \langle \vec{p}_3, \vec{p}_4 | :\phi^4(x): | \vec{p}_1, \vec{p}_2 \rangle$

$$\langle 0 | a(\vec{p}_3)\, a(\vec{p}_4) :\phi^4(x): a^\dagger(\vec{p}_1)\, a^\dagger(\vec{p}_2) | 0 \rangle$$

$$\phi(x) = \underbrace{\int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}} \left[a(\vec{p})\, e^{-ip\cdot x} \right]}_{\equiv \phi^{(+)}(x)} + \underbrace{\int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}}\, a^\dagger(\vec{p})\, e^{ip\cdot x}}_{\equiv \phi^{(-)}(x)}$$

$$:\phi^4(x): \;\rightarrow\; \underbrace{\frac{4!}{2!2!}}_{\equiv \binom{4}{2}} \underbrace{\left\{ \phi^{(-)}(x) \right\}^2 \left\{ \phi^{(+)}(x) \right\}^2}_{=}$$

2 LOWERING AND 2 CREATION OPERATORS

(49)

NOW $\langle 0| a(\vec{p}_3)\, a(\vec{p}_4) \{\phi^{(-)}(x)\}^2 \{\phi^{(+)}(x)\}^2 a^+(\vec{p}_1)\, a^+(\vec{p}_2)|0\rangle$

$$= \int dp\, dp'\, dp''\, dp''' \langle 0| a(\vec{p}_3)\, a(\vec{p}_4)\, a^+(\vec{p})\, a^+(\vec{p}\,')\, a(\vec{p}\,'')\, a(\vec{p}\,''')\, a^+(\vec{p}_1)\, a^+(\vec{p}_2)|0\rangle$$
$$\times\, e^{i(p+p'-p''-p''')\cdot x}$$

$\boxed{\text{USING } dp \equiv \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}}}$

CONSIDER:

$$a(\vec{p}\,'')\, a(\vec{p}\,''')\, a^+(\vec{p}_1)\, a^+(\vec{p}_2)|0\rangle$$

$$= a(\vec{p}\,'')\left[a^+(\vec{p}_1)\, a(\vec{p}\,''') + (2\pi)^3 2E_{\vec{p}_1} \delta^{(3)}(\vec{p}_1 - \vec{p}\,''')\right] a^+(\vec{p}_2)|0\rangle$$

$$= (2\pi)^6\, 2E_{\vec{p}_1}\, 2E_{\vec{p}_2} \left[\delta^{(3)}(\vec{p}\,'' - \vec{p}_1)\, \delta^{(3)}(\vec{p}\,''' - \vec{p}_2) + 1 \leftrightarrow 2 \right]|0\rangle$$

SIMILARLY: $\langle 0| a(\vec{p}_3)\, a(\vec{p}_4)\, a^+(\vec{p})\, a^+(\vec{p}\,') =$

$$= \langle 0| \left[(2\pi)^6\, 2E_{\vec{p}_3}\, 2E_{\vec{p}_4}\, \delta^{(3)}(\vec{p} - \vec{p}_3)\, \delta^{(4)}(\vec{p}\,' - \vec{p}_4) + 3 \leftrightarrow 4 \right]$$

$$= \int d^3\vec{p}\, d^3\vec{p}\,'\, d^3\vec{p}\,''\, d^3\vec{p}\,'''\, e^{i(p+p'-p''-p''')\cdot x} \langle 0|0\rangle$$
$$\times \left[\delta^{(3)}(\vec{p} - \vec{p}_3)\, \delta^{(3)}(\vec{p}\,' - \vec{p}_4)\, \delta^{(3)}(\vec{p}\,'' - \vec{p}_1)\, \delta^{(3)}(\vec{p}\,''' - \vec{p}_2) + 3 \text{ SIMILAR TERMS} \right]$$

$$= 4\, e^{i x\cdot(p_3 + p_4 - p_1 - p_2)}$$

4-vectors

50

ALLTOGETHER:

$$\frac{(-i\lambda)}{4!}\underbrace{\binom{4}{2}}_{\frac{4!}{2!2!}=6}\cdot 4\int d^4x\, e^{ix\cdot(p_3+p_4-p_1-p_2)}$$

$$= (-i\lambda)(2\pi)^4\delta^{(4)}(\vec{p}_3+\vec{p}_4-\vec{p}_1-\vec{p}_2)$$

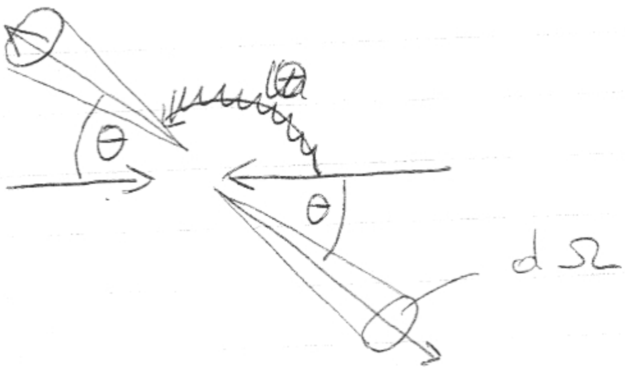
$$\Rightarrow \quad T = -i\lambda \qquad \text{(TO FIRST ORDER)}$$

T ↑ SCATTERING AMPLITUDE

- THE DIFFERENTIAL CROSS SECTION IS (WITHOUT PROOF)

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}|T|^2 = \frac{\lambda^2}{64\pi^2 s} \qquad s=(p_1+p_2)^2$$

$$d\Omega = \sin\theta\, d\theta\, d\phi$$



SECOND ORDER

$$\frac{1}{2}\frac{(-i\lambda)^2}{(4!)^2}\int d^4x\, d^4y \, \langle 0|\, a(\vec{p}_3)\, a(\vec{p}_4)\; T :\phi^4(x)::\phi^4(y):\; a^+(\vec{p}_1)\, a^+(\vec{p}_2)\,|0\rangle$$

$T:\phi^4(x)::\phi^4(y):$ contains terms like

$$\underbrace{(i\Delta_F(x-y))^2}_{\text{FROM 2 WICK CONTRACTIONS}} :\phi^2(x)\,\phi^2(y):$$

ANY OF THE 4 FIELDS CAN COMBINE WITH THE a OR a^+ OPERATORS AS IN THE FIRST ORDER EXAMPLE

ONE PARTICULAR TERM WILL BE:

$$\sim (-i\lambda)^2 N \int d^4x\, d^4y\; e^{iy\cdot(p_3+p_4)}\; e^{-ix\cdot(p_1+p_2)}$$

N DENOTES (A) COMBINATORIAL FACTOR(S) WHICH WE DO NOT KEEP TRACK OFF!

$$\times \frac{i^2}{(2\pi)^8}\int d^4p\, d^4p' \; \frac{e^{-ip\cdot(x-y)}\; e^{-ip'\cdot(x-y)}}{(p^2-m^2+i\epsilon)(p'^2-m^2+i\epsilon)}$$

$$(-i\lambda)^2 N \int d^4p\, d^4p' \; \frac{i^2}{(p^2-m^2+i\epsilon)(p'^2-m^2+i\epsilon)}$$

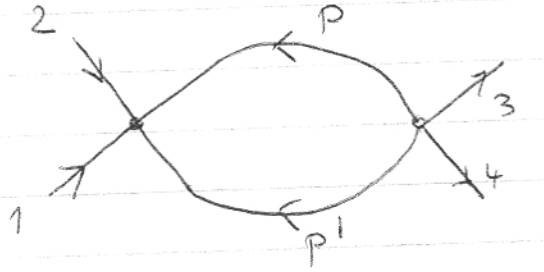
$$\underbrace{\int \frac{d^4x}{(2\pi)^4}\; e^{-ix\cdot(p_1+p_2+p+p')}}_{\delta^{(4)}(p_1+p_2+p+p')}\; \underbrace{\int \frac{d^4y}{(2\pi)^4}\; e^{iy\cdot(p_3+p_4+p+p')}}_{\delta^{(4)}(p_3+p_4+p+p')}$$

p' - INTEGRATION SETS $p' = -p - p_3 - p_4$

$$= (-i\lambda^2) N \int d^4p \; \frac{i}{(p^2-m^2+i\epsilon)} \times \frac{i}{(p+p_3+p_4)^2 - m^2 + i\epsilon}$$

$$\times\; \delta^{(4)}(p_1+p_2-p_3-p_4)$$

(FEYNMAN) GRAPHICALLY



1-LOOP DIAGRAM
(INTEGRAL OVER p DIVERGES!
→ REGULARISATION
→ RENORMALISATION)

- EACH VERTEX GIVES A FACTOR OF $(-i\lambda) \times (2\pi)^4 \delta^{(4)}$ (MOMENTUM CONSERVATION)
$= (-i\lambda)(2\pi)^4 \delta^{(4)}$ (SUM OF INCOMING − OUTGOING MOMENTA)
- EACH INTERNAL LINE GIVES A FACTOR

$$\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon}$$

- THESE ARE THE FEYNMAN RULES FOR THE $\lambda\phi^4$-THEORY AND COMPLETELY DEFINE PERTURBATION THEORY OF THE ϕ^4-THEORY!

ANOTHER SECOND ORDER EXAMPLE

SCATTERING OF 3 PARTICLES $\to$ 3 PARTICLES

$$\frac{1}{2}\frac{(-i\lambda)^2}{(4!)^2}\int d^4x\, d^4y \,\langle \vec{p}_4,\vec{p}_5,\vec{p}_6 |\, T :\phi^4(x)::\phi^4(y): |\vec{p}_1,\vec{p}_2,\vec{p}_3\rangle$$

IN THIS CASE WE KEEP TERMS OF THE FOLLOWING FORM FROM WICK'S THEOREM

$$i\Delta_F(x-y) :\phi^3(x)\,\phi^3(y):$$

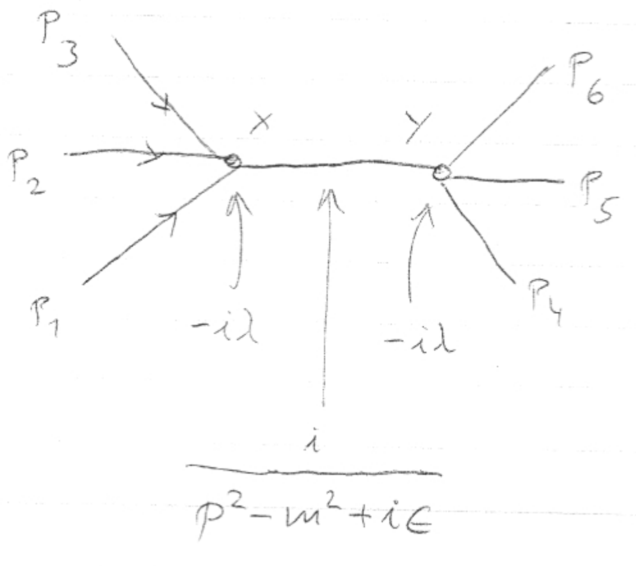
SO WE GET (THIS IS ONE PARTICULAR CONTRIBUTION)
(AGAIN WE IGNORE COMBINATORIAL FACTORS)

$$\simeq (-i\lambda)^2 \int \frac{d^4p}{(2\pi)^4}\int d^4x\, d^4y \; e^{ix\cdot(p_4+p_5+p_6)}\, e^{-iy\cdot(p_1+p_2+p_3)} \times \frac{i\, e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}$$

$$=(-i\lambda^2)\int \frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2-m^2+i\epsilon}\,(2\pi)^4\delta^{(4)}(p_4+p_5+p_6-p) \times (2\pi)^4\delta^{(4)}(p_1+p_2+p_3-p)$$

$$= i\,(2\pi)^4\delta^{(4)}(p_1+p_2+p_3-p_4-p_5-p_6)\,\frac{(-i\lambda)^2}{p^2-m^2+i\epsilon}$$

WITH $p=p_1+p_2+p_3$



FEYNMAN DIAGRAM

- FEYNMAN RULES GIVE THE SAME ANSWER!

FEYNMAN RULES OF QED

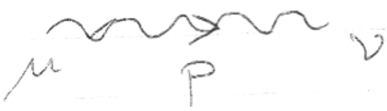
$$\mathcal{L}_{QED} = \underbrace{-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}} + \bar{\psi}(i\not{\partial} - m)\psi + \underbrace{q:\bar{\psi}\not{A}\psi:}_{\mathcal{L}_I}$$

"$\partial_\mu A^\mu = 0$" GAUGE

$$-\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu$$

- PROPAGATOR:

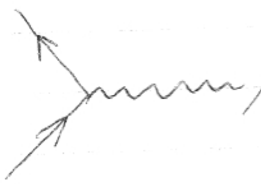
$$D_{\mu\nu}(p) = -i\frac{g_{\mu\nu}}{p^2 + i\epsilon}$$



- PROPAGATOR:

$$\frac{i(\not{p} + m)}{p^2\mathbb{1} - m^2 + i\epsilon}$$

- VERTEX



$\mu \equiv iq\gamma^\mu = -ie\gamma^\mu$ (FOR ELECTRON)

- INCOMING/OUTGOING ELECTRONS $U(p_i, s_i) / \bar{U}(p_f, s_f)$

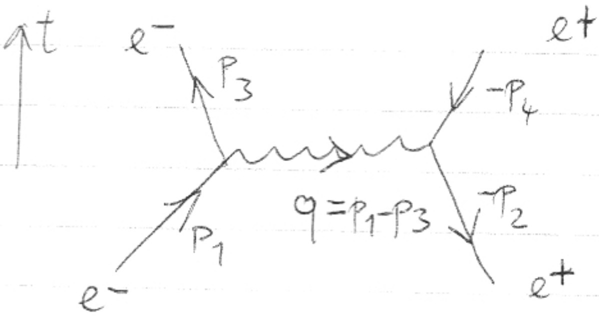
INCOMING/OUTGOING POSITRONS $V(p_i, s_i) / \bar{V}(p_f, s_f)$

(POSITRONS ARE DRAWN AS NEG. ENERGY ELECTRONS RUNNING BACKWARDS IN TIME)

EXAMPLES OF LOWEST ORDER SCATTERING AMPLITUDES IN QED:

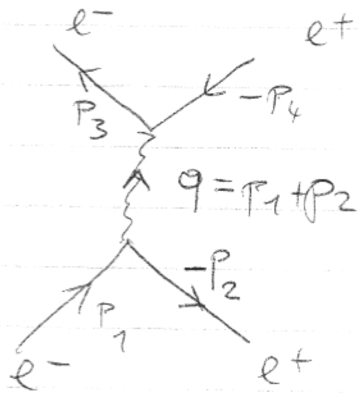
1) $e^- e^+ \rightarrow e^- e^+$ (Bhobha Scattering) LEP

2 FEYNMAN DIAGRAMS CONTRIBUTE



$\ominus$

FERMI STATISTICS



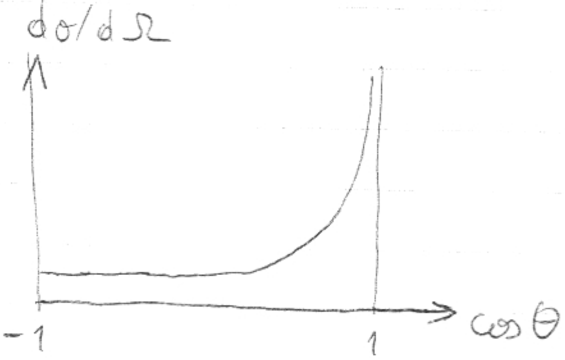
$$\Rightarrow (-ie)^2 \Big[\big(\bar U(p_3)\,\gamma^\mu\, U(p_1)\big) \frac{g_{\mu\nu}(-i)}{(p_1-p_3)^2+i\epsilon} \big(\bar V(p_2)\,\gamma^\nu\, V(p_4)\big)$$

$$- \big(\bar V(p_2)\,\gamma^\mu\, U(p_1)\big) \frac{g_{\mu\nu}(-i)}{(p_1+p_2)^2+i\epsilon} \big(\bar U(p_3)\,\gamma^\nu\, V(p_4)\big) \Big]$$

ANTI-SYMMETRY UNDER: $U(p_1) \Leftrightarrow V(p_4)$ & $\bar U(p_3) \leftrightarrow \bar V(p_2)$

- FOR $E \gg m$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2E_{CM}^2}\left[\frac{1+\cos^4\frac{\theta}{2}}{\sin^4\frac{\theta}{2}} + \frac{1+\cos^2\theta}{2} - \underbrace{\frac{2\cos^4\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}}_{\text{INTERFERENCE TERM}}\right]$$



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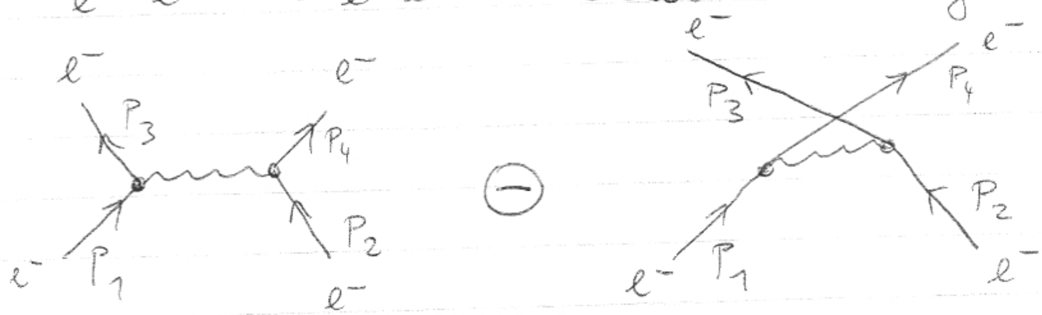
2) $e^- e^- \to e^- e^-$ Möller Scattering



Identical spin $\frac{1}{2}$ Particles

$$(-ie)^2\Big[\left(\bar{U}(p_3)\gamma^\mu U(p_1)\right)\frac{-ig_{\mu\nu}}{(p_1-p_3)^2+i\epsilon}\left(\bar{U}(p_4)\gamma^\nu U(p_2)\right)$$

$$-\left(\bar{U}(p_4)\gamma^\mu U(p_1)\right)\frac{-ig_{\mu\nu}}{(p_1-p_4)^2+i\epsilon}\left(\bar{U}(p_3)\gamma^\nu U(p_2)\right)\Big]$$

ANTI-SYMMETRY : $1 \leftrightarrow 2$
$3 \leftrightarrow 4$

- $E \gg m$

FORWARD SCATTERING DOMINATED ↓ DOMINATES IN BACKWARD SCATT. ↓

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2E_{CM}^2}\left[\frac{1+\cos^4\frac{\theta}{2}}{\sin^4\theta/2} + \frac{2}{\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}} + \frac{1+\sin^4\frac{\theta}{2}}{\cos^4\theta/2}\right]$$

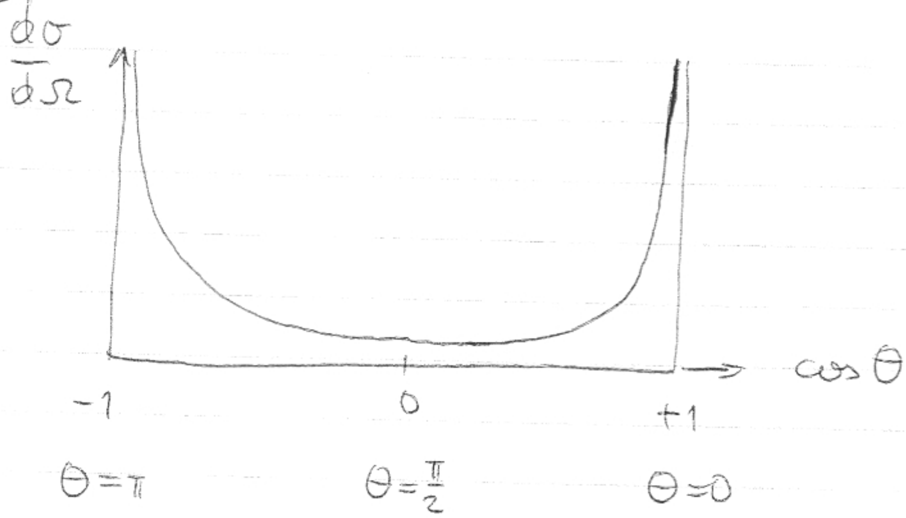
↑ INTERFERENCE TERM

FOR BOSONS WE WOULD GET THERE A MINUS SIGN

SYMMETRY $\theta \to \pi - \theta$

EXPERIMENTALIST CANNOT TELL IF e^- WAS SCATTERED FORWARD OF BACKWARDS.

56



$\frac{d\sigma}{d\Omega}$
$\cos\theta$
-1
0
$+1$
$\theta=\pi$
$\theta=\frac{\pi}{2}$
$\theta=0$