USEFUL DEFINITIONS

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INERTIAL REFERENCE FRAME

⇒ <u>REFERENCE FRAME</u>: consist in an origin in space, three orthogonal axes, and a clock

⇒ <u>INERTIAL REFERENCE FRAMES</u> is a subset of all the possible reference frames, where an isolated, non rotating and unaccelerated body moves uniformly on a straight line.

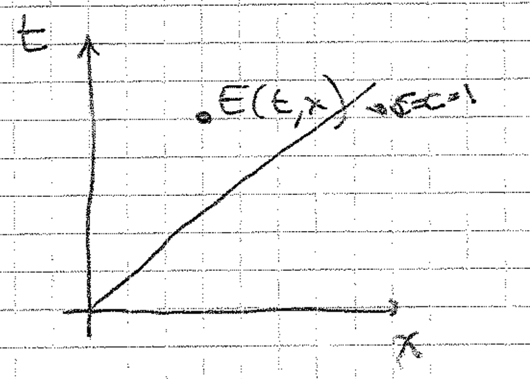
⇒ <u>EVENT</u>: is a single point in space together with a single point in time: it's denoted by 4 real numbers:

a real number t giving it's Newtonian time

a triplet (x, y, z) giving its position in space

$$(t, x, y, z) \rightarrow \text{EVENT (a POINT) IN SPACETIME}$$

To visualize it we can just drop two of the spatial direction



The trajectory of a material point in spacetime is its WORLDLINE

Unaccelerated bodies have straight worldline

RECAP OF SPECIAL RELATIVITY

The equivalence principle can be rephrased by saying that in the neighborhood of any point we can find a reference frame where the laws of physics take the form dictated by special relativity.

Before investigating in detail how this happen, let's do a recap of special rel.

The postulates at the basis of special relativity say that

1. the fundamental laws of physics have the same form in any inertial reference frame.

2. the speed of light in the vacuum is a universal constant.

Keeping in mind that in G.R. we will deal with "locally inertial reference frames" (ie freely falling frames), we can consider two inertial reference frames.

$$\underset{(t,x,y,z)}{K} \qquad \underset{(t',x',y',z')}{K'}$$

such that the axis are orientated parallel

and let's suppose K' and K are moving with a relative velocity $\vec{v}$ such that THE AXES REMAINS PARALLEL, which is directed for example along the direction $\hat{x}$

ADDENDUM : NATURAL UNITS

What is the meaning of imposing $c = 1$?

In ordinary units, c is a velocity, thus it has dimensions:

$$[c] = \frac{[L]}{[T]} \qquad \text{i.e. in the S.I } \frac{m}{s}$$

Imposing natural units means imposing

$$c = 1$$

which does not only change the "numerical" value of c, but it makes it a DIMENSIONLESS QUANTITY!

THIS implies that in natural units the following relation between dimensions holds

$$[L] = [T]$$

length and time are measured with the same unit!

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Then the two reference frames are related by a Lorentz transformation (boost)

$$\begin{cases} t' = \dfrac{t - v x}{\sqrt{1-v^2}} \\ x' = \dfrac{x - vt}{\sqrt{1-v^2}} \\ y' = y \\ z' = z \end{cases}$$

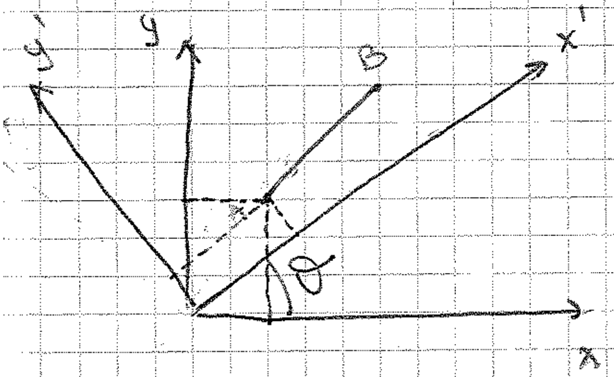
$\Rightarrow$ these are written in natural units: $c = 1$

it means that $[L] = [T]$

$\downarrow$

this can be got assuming that at the instant $t = t' = 0$ a flash of light is emitted by the common origin, and by requiring that the propagation of the spherical wave front is compatible with homogeneity and isotropy of space and homogeneity of time.

BUT we can also consider other transformations relating K and K'; let's consider a "proper" rotation in space, for example about the $\hat{z}$ axis



The coordinates of a point P

$$\begin{cases} t' = t \\ x' = x\cos\theta + y\sin\theta \\ y' = -x\sin\theta + y\cos\theta \\ z' = z \end{cases}$$

We can put this transformation in a matrix form:

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

This is a rigid rotation, which means that, if we consider two points, their coordinates in K and K' are different, however their distance is constant (i.e the length of the segment).

This is (squared)

$$\Delta s^2 = \Delta x'^2 + \Delta y'^2 = \Delta x^2 + \Delta y^2$$

Thus the quantity

$$(\Delta x)^2 + (\Delta y)^2$$

is invariant under rotation.

Analogously, if we had done a rotation in three dimensions, the invariant quantity would have been the distance between two points in 3 dimensions, which is:

$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 = \Delta x'^2 + \Delta y'^2 + \Delta z'^2$$

and, if we consider two points infinitesimally separated
$P: (x_P, y_P, z_P)$ and $Q /\, x_Q = x_P + dx,\ y_Q = y_P + dy,\ z_Q = z_P + dz$

$$ds^2 = dx^2 + dy^2 + dz^2 = dx'^2 + dy'^2 + dz'^2$$

invariant under spatial rotations

Thus now we can wonder if there is a quantity which the Lorentz transformations leave invariant.

Lorentz transformations act in spacetime, thus the invariant quantity we are looking for must involve x, y, z and t.

Let's start reparametrising the Lorentz transformations in a convenient way. Let's introduce the quantity called

$$\text{RAPIDITY} : \xi \quad / \quad v = \tanh\xi$$

HOW DOES THE LORENTZ TRANSFORMATIONS LOOK IN TERMS OF ξ?

$$\Rightarrow 1 - v^2 = 1 - \tanh^2\xi = 1 - \frac{\sinh^2\xi}{\cosh^2\xi} = \frac{\cosh^2\xi - \sinh^2\xi}{\cosh^2\xi}$$

BUT $\cosh^2\xi - \sinh^2\xi = 1$

Thus: $\sqrt{1 - v^2} = \dfrac{1}{\cosh\xi}$

Then we have

$$t' = \cosh\xi \left(t - \tanh\xi\, x \right) = \cosh\xi\, t - \sinh\xi\, x$$

$$x' = \cosh\xi \left(x - \tanh\xi\, t \right) = -\sinh\xi\, t + \cosh\xi\, x$$

$$y' = y$$

$$z' = z$$

This transformation can be put in matrix form as:

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\xi & -\sinh\xi & 0 & 0 \\ -\sinh\xi & +\cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

With this parametrization we see that the "shape" of this transf. is very similar to the spatial rotation, with the difference that it acts on spacetime, thus we may wonder if, in analogy to what we saw earlier, the invariant quantity is an invariant distance between two points in spacetime. (*)(*)

HOW DO WE DEFINE IT?

We can consider two ~~points~~ events in K and K', and we see that we consider the path of a pulse of light in K and K' and we will find that the combination (use $t \to -it$)

$$(\Delta s)^2 = -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

is the same in both frames, namely

$$-(\Delta t')^2 + (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2 = -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

thus

$$\boxed{(\Delta s)^2 \text{ is invariant under a Lorentz transformation.}}$$

this is THE DISTANCE BETWEEN POINTS IN SPACETIME (EVENTS) WHICH SPECIFY THE SPACETIME GEOMETRY

THE RELATION BETWEEN ROTATIONS AND LORENTZ BOOSTS IS DEEPER (MORE THAN JUST A SIMILARITY)

→ let's consider a rotation but let's perform it in t and x, and let's replace

$$\theta \rightarrow i\xi$$

$$t' = t\cos i\xi + x\sin i\xi$$

$$x' = -t\sin i\xi + x\cos i\xi$$

BUT $\sin i\xi = +i\sinh\xi$, $\cos i\xi = \cosh\xi$

$$\Rightarrow \quad t' = t\cosh\xi + ix\sinh\xi$$

$$x' = -ti\sinh\xi + x\cosh\xi$$

Now let's rotate t to imaginary axis: $t \rightarrow -it$

$$-\cancel{i}t' = -\cancel{i}t\cosh\xi + \cancel{i}x\sinh\xi$$

$$x' = -(-it)i\sinh\xi + x\cosh\xi \quad \Rightarrow$$

$$\Rightarrow \begin{cases} t' = t\cosh\xi - x\sinh\xi \\ x' = -t\sinh\xi + x\cosh\xi \\ y' = y \end{cases}$$

GROUP $SO(1,3)$ ↓ LORENTZ GROUP

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if we consider $\Delta t = t - t_0$, $\Delta x = x - x_0$, etc, and we assume $t_0 = x_0$, in 2d we have

$$-t'^2 + x'^2 = -t^2 + x^2$$

Let's prove it:

$$-t'^2 + x'^2 = -\left(\cosh\xi\, t - \sinh\xi\, x\right)^2 + \left(-\sinh\xi\, t + \cosh\xi\, x\right)^2 =$$

$$= -\cosh^2\xi\, t^2 - \sinh^2\xi\, x^2 + \cancel{2\sinh\xi\cosh\xi\, t x} +$$

$$+ \sinh^2\xi\, t^2 + \cosh^2\xi\, x^2 - \cancel{2\cosh\xi\sinh\xi\, t x} =$$

$$= -t^2\left(\cosh^2\xi - \sinh^2\xi\right) + x^2\left(\cosh^2\xi - \sinh^2\xi\right) =$$

$$= -t^2 + x^2$$

thus, let's recapitulate:

$(t, \vec{x})$ is a point in spacetime (event)

→ the distance between two events is invariant under Lorentz transformations.

We can consider two points infinitesimally separated, the invariant distance is

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 =$$

$$= -dt'^2 + dx'^2 + dy'^2 + dz^2$$

~~here spacelike~~

Notice that this transformation can be seen as linear transform. on the matrix

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Let's focus just on 2 dimensions

$$\begin{pmatrix} \cosh\xi & -\sinh\xi \\ \sinh\xi & +\cosh\xi \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\xi & -\sinh\xi \\ -\sinh\xi & +\cosh\xi \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$$

What we have just done is finding a way to describe the intrinsic geometry of spacetime

Defining the infinitesimal distance between two events we we can actually define a line element

$$ds = \sqrt{-dt^2 + dx^2}$$

which is actually giving the distance between two points in terms of ~~the~~ specific set of coordinate, thus we are specifying ~~the~~ GEOMETRY OF A GIVEN SPACE.

In fact, STARTING FROM THE infinitesimal line element, we can build up the distance among different points upon integration, and then use differential and integral calculus to reduce all geometry to the specification of the distance between two adjacent points.

Thus:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

defines the geometry of 4-dimensional spacetime, and it is the starting point of general relativity.

The space it describes is 4-d Minkowsky space, it is a flat (non curved) spacetime and, because of the - sign, it's non euclidean.

END OF W2/L1
(04/10)

In Minkowsky space we can introduce a new notation and describe points (events) with 4-vectors

$$x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

μ is an index which labels the different coordinates

we define then the square length of the four vector x^μ, which is just the squared associated displacement $\Delta s^2 = -t^2 + x^2 \dots = \sum_{\mu,\nu=} \eta_{\mu\nu} x^\mu x^\nu$ (1)
and we define the scalar product between two 4-vectors as

AFTER!

(3) $\boxed{\text{SCALAR PRODUCT } x \cdot y}$ (*) $= \sum_{\mu,\nu} \eta_{\mu\nu} x^\nu y^\mu \Rightarrow$

$= \sum_{\mu,\nu} \eta_{\mu\nu} y^\mu x^\nu$

$$\eta_{\mu\nu} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\eta_{\mu\nu}$ is the METRIC, namely the way you compute distance between two points in a given coordinate set

So: dx^μ (infinitesimal separation between two points

Then we can write

(2)

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dx^0 dx^0 + dx^1 dx^1 + dx^2 dx^2 + dx^3 dx^3$$

(*) we won't use this notation, but we will use

$$x_\mu x^\mu \quad \text{where} \quad x_\mu = \eta_{\mu\nu} x^\nu$$

we use the MINKOWSKY METRIC
TO LOWER THE INDEX!

Thus: $x^\mu = (t, \vec{x})$

x_μ has components:

$$x_0 = \eta_{0\nu} x^\nu = \eta_{00} x^0 + \eta_{01} x^1 + \eta_{02} x^2 + \eta_{03} x^3 = -t$$

$$x_1 = \eta_{1\nu} x^\nu = \eta_{10} x^0 + \eta_{11} x^1 + \eta_{12} x^2 + \eta_{13} x^3 = x$$

can obtain x_2 and x_3 in the same way!

Thus $x_\mu = (-t, x, y, z) = (-t, \vec{x})$

and $x_\mu x^\mu = x_0 x^0 + x_1 x^1 + x_2 x^2 + x_3 x^3 =$

$$= -t^2 + x^2 + y^2 + z^2$$

NB: this holds for any vector!

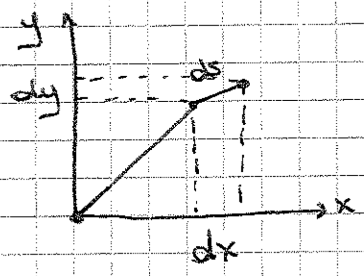
$$v^\mu = \begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix} \Rightarrow v_\mu = \eta_{\mu\nu} v^\nu = (-v^0, v^1, v^2, v^3)$$

$$\Rightarrow u^\mu = \begin{pmatrix} u^0 \\ u^1 \\ u^2 \\ u^3 \end{pmatrix} \Rightarrow u_\mu = \eta_{\mu\nu} u^\nu = (-u^0, u^1, u^2, u^3)$$

$\Rightarrow$ we can compute $v_\mu v^\mu$, $u_\mu u^\mu$, $v_\mu u^\mu$

We can also consider the inverse metric: $(\eta_{\mu\nu})^{-1} \equiv \eta^{\mu\nu}$
and use it to raise the indices: $x^\mu = \eta^{\mu\nu} x_\nu$, $\Rightarrow \eta_{\mu\nu}\eta^{\nu\rho} = \delta_\mu^\rho$

EXAMPLE: Let's consider the 2-dimensional euclidean plane

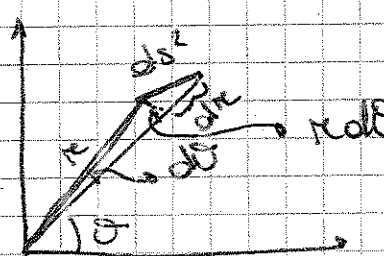


for infinitesimal distances

$$ds^2 = dx^2 + dy^2 = (dx\ dy)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}$$

so here the metric is $\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Now let's suppose we want to describe THE SAME GEOMETRY in a different coordinate set, for example polar



here the coordinates are (r, θ)

Thus, the infinitesimal distance ds^2 is

(*) $$ds^2 = dr^2 + r^2 d\theta^2 = (dr\ d\theta)\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}\begin{pmatrix} dr \\ d\theta \end{pmatrix}$$

Thus in this coordinate set the metric is

$$\eta_{\mu\nu} \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

$\downarrow$

it depends on the choosen coordinate frame!

(*) $x = r\cos\theta$, $y = r\sin\theta$, then $dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta = \cos\theta\, dr - \sin\theta\, d\theta$
and $dy = \sin\theta\, dr + \cos\theta\, d\theta$.
Then just compute dx^2 and dy^2 and replace it in ds^2