THE SCHWARZCHILD METRIC

Now we have the Einstein equations,

→ set of equations describing the curved geometry in presence of matter and energy density.

WHAT IS THE "SHAPE" OF SPACETIME IN THE PRESENCE OF A GIVEN MATTER AND ENERGY DISTR.!

Simplest solution: SPHERICALLY SYMMETRIC GRAVITATIONAL FIELD.

This is, for example, the gravitational field created by the sun, around which the planets move. OR BY THE Earth itself.

→ WE ARE INTERESTED IN A SOLUTION OF THE VACUUM EINSTEIN EQUATIONS.

$T_{\mu\nu} \neq 0$

$\Downarrow$

$G_{\mu\nu} = 8\pi G T_{\mu\nu}$ hold

$T_{\mu\nu} = 0$

$\Downarrow$

EINST. EQ. HERE ARE

$R_{\mu\nu} = 0$

the form of the metric here is a consequence of the symmetry of the problem

We will say something about

→ properties of the gravitational field around such a body ⇒ spherical symmetry

→ motion of test particle (both along timelike and null geodesics) in the neighborhood of such an object.

What we are going to learn about G.R. in such a "purely relativistic" background, the solution of G.R. that we are going to study, will be applicable to other HIGHLY RELATIVISTIC objects: BLACK HOLES.

In G.R. it exists a UNIQUE SOLUTION OF THE VACUUM EINSTEIN EQUATIONS WITH SPHERICAL SYMMETRY

⇒ Convenient to use spherical coordinates

$$t, r, \theta, \phi$$

⇒ THIS IS THE SCHWARZCHILD METRIC

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 +$$

$$+ r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

HOW DO WE DERIVE THE SCHWARZCHILD METRIC?

$\Rightarrow$ SOLUTION OF $R_{\mu\nu} = 0$

which is spherically symmetric.

To find it we can write an ansatz for the metric, where we generalize the flat metric multiplying each term for arbitrary functions of r (no θ and ϕ because of spherical symmetry). Thus:

$$ds^2_{Mink} = -dt^2 + dr^2 + \underbrace{r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2}$$

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 \underbrace{\left(d\theta^2 + \sin^2\theta \, d\phi^2\right)}$$

EXPONENTIALS TO PRESERVE SIGNATURE WHEN $r \to \infty$

NO COEFFICIENT HERE, BECAUSE IT CAN ALWAYS BE REABSORBED IN THE DEFINITION OF r

$d\Omega^2 \Rightarrow$ this term is not modified because we want the spheres around the source to be perfectly round!

THUS THE EINSTEIN EQUATIONS CAN BE REWRITTEN AS DIFFERENTIAL EQUATIONS FOR $\alpha(r)$, $\beta(r)$

Thus we can compute the Christoffel symbols:

$$\Gamma^{t}_{rt} = \partial_r \alpha(r) \, , \quad \Gamma^{r}_{rr} = \partial_r \beta(r) \, , \quad \text{etc...}$$

out of the Christoffel symbols we compute the components of the Riemann tensor

$$R^{t}{}_{rtr} = \partial_r \alpha(r)\, \partial_r \beta(r) - \partial_r^2 \alpha(r) - \left(\partial_r \alpha(r)\right)^2$$

$$R^{\theta}{}_{\phi\theta\phi} = \left(1 - e^{-2\beta(r)}\right) \sin^2\theta$$

$$\vdots$$

etc.

and then we contract it with the metric to get the components of the Ricci tensor. Thus the EINSTEIN EQUATIONS ARE

$$R_{tt} = e^{2(\alpha-\beta)}\left[\partial_r^2 \alpha + (\partial_r \alpha)^2 - \partial_r \alpha\, \partial_r \beta + \frac{2}{r}\partial_r \alpha\right] = 0$$

$$R_{rr} = -\partial_r^2 \alpha - (\partial_r \alpha)^2 + \partial_r \alpha\, \partial_r \beta + \frac{2}{r}\partial_r \beta = 0$$

$$R_{\theta\theta} = e^{-2\beta}\left[r\left(\partial_r \beta - \partial_r \alpha\right) - 1\right] + 1 = 0$$

$$R_{\phi\phi} = \sin^2\theta \; R_{\theta\theta} = 0$$

We can combine these equation together, eg. if you do ① c + ② = 0

$$\Rightarrow \quad \frac{2}{r}\left(\partial_r \alpha + \partial_r \beta\right) = 0$$

$$\Rightarrow \quad \alpha(r) = -\beta(r) + \underbrace{\text{constant}}_{= 0}$$

$$\alpha(r) = -\beta(r)$$

Then $R_{\theta\theta} = 0 \quad \Rightarrow \quad \partial_r \left(r e^{2\alpha(r)} \right) = 1$

$$\Rightarrow \quad e^{2\alpha} = 1 - \frac{R_s}{r}$$

where R_s is a constant of integration. From this

$$ds^2 = -\left(1 - \frac{R_s}{r}\right) dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

⇒ WHAT IS THE MEANING OF THE INTEGRATION CONSTANT ?

To give it a meaning let's consider the metric that we wrote in the weak field static limit

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

and we found $g_{00} = g_{tt} = -1 + 2\phi = -\left(1 - \frac{2GM}{r}\right)$

We can require that the Schwarzschild metric coincide with the weak field metric when we are far from the source, eg $r \to \infty$. $(r >> 2GM)$.

In this case, we see that the tt term of the metric is already equal to the weak field case if:

$$R_S = 2GM \qquad \left(\begin{array}{l} \text{if } c \neq 1 \\ R_S = 2GM/c^2 \end{array}\right)$$

and M is the Newtonian mass of the source of the gravitational field, ie the mass that we would measure if we were studying the gravitational field sitting far from its source. With this we get

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

$1+2\phi$ $\quad$ $1-2\phi$

NEWTONIAN LIMIT : $ds_N^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 + 2\frac{GM}{r}\right)dr^2 + r^2 d\Omega^2$

SINGULARITIES

⇒ the coefficients of the metric becomes infinity (i.e. "meaningless") at

$$r = 0 \qquad r = R_S = 2GM$$

↳ Schwarzschild radius.

What's happening there?

⇒ We saw some weeks ago that this might just be the sign that something is wrong with our choice of coordinates, and not with spacetime geometry itself. (COORDINATE SINGULARITY)

EG: we can consider the two dimens metric

$$ds^2 = \frac{1}{t^4}\left(dt^2 + t^2 d\theta^2\right)$$

⇒ diverges for $t = 0$, but if we cons. $r = \frac{1}{t}$ the line element become

$$ds^2 = dr^2 + r^2 d\theta^2 \quad \Rightarrow$$ IT'S JUST THE PLANE!

and $t = 0$ is the limit for $r \to \infty$, where nothing bad happens at all; (t, θ) was just not the good set of coordinates for describing that point.

HOW DO WE IDENTIFY COORDINATE SINGULARITIES?

⇒ Thus let's look for some coordinate independent quantities which can tell us if the geometry at $r = 0$ and $r = 2GM$ goes out of control.

These must be scalars, i.e. object which are invariant under coordinate transformation ⇒

which means that their (possibly) bad behaviour does not depend just on a bad coordinate choice.

These for example can be

→ Ricci scalar $\quad R = R_{\mu\nu} g^{\mu\nu}$

but also other, more complicated things such

$$\underbrace{R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}}, \qquad (*)$$

$$\underbrace{R_{\mu\nu\rho\sigma} R^{\mu\nu\alpha\beta} R_{\alpha\beta}{}^{\rho\sigma}}, \quad \text{etc.}$$

→ if some of these scalars goes to infinity in those specific points, then we really have a curvature singularity.

Let's consider for example (*). For the Schwarzschild metric this is

$$R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{48 G^2 M^2}{r^6}$$

Thus:

→ $r = 0$: is a true curvature singularity

→ $r = 2GM$:

$$R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}\Big|_{r=2GM} = \frac{3}{4 G^4 M^4}$$

well defined finite value

none of the others curvature invariants blow up here

→ Actually the Schwartzchild radius has a physical meaning, WE will see that it is the EVENT HORIZON OF A BLACK HOLE, this of course where this distance is still in the vacuum.

⇝ For a regular object, like the Sun or the Earth,

$$R_S \ll R_\odot \quad , \quad \text{or} \quad R_S \ll R_\oplus$$

Thus it falls inside the object which generates the field. in a region where the Schwartzchild metric is no longer the right description of the geometry:

ETC. for the Sun and the Earth

$$R_\odot \approx 7 \times 10^5 \text{ Km} \quad , \quad \frac{2GM_\odot}{c^2} \sim 2.9 \text{ Km}.$$

$$R_\oplus \approx 6371 \text{ Km} \qquad \frac{2GM_\oplus}{c^2} \sim 0{,}88 \text{ cm}$$

LIMITS

LIMITS ON THE PARAMETER M:

$\rightarrow$ if $M \rightarrow 0$ $\rightarrow$ $ds^2_{Schw} \longrightarrow ds^2_{Mink.}$

expected: no source for the grav. field if $M \rightarrow 0$

LARGE DISTANCE BEHAVIOUR

Notice that as $r \rightarrow \infty$, the metric tends to the Minkowsky result.

This phenomenon is called ASYMPTOTIC FLATNESS and it is a "consequence" of the Schwarzchild metric, not something that we required a priori.

It means that at large distances we can really identify t and r respectively with the time and rad. distance of special relativity.